AIM Qualifying Exam: Advanced Calculus and Complex Variables

Jan 2024

There are five (5) problems in this test, each worth 20 points.

For the most part, there is sufficient room in this booklet for all your work. However, if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

1. Let

$$\alpha_1, \alpha_2, \ldots$$

be an enumeration of rational numbers in the interval $[0, 1)$. Define $f : [0, 1] \to \mathbb{R}$ as

$$f(x) = \sum_{\left\{ j \middle| \alpha_j < x \right\}} \frac{1}{2^j}$$

for $0 < x \leq 1$ and $f(x) = 0$ for $x = 0$.

a) Prove that f is discontinuous at rational x with $x < 1$.

b) Prove that f is continuous at irrational x.

2. Let γ_n be the path

$$\gamma_n(t) = \left(t, \frac{\sin 2\pi t}{n}\right), \quad 0 \leq t \leq 1,$$

and let γ be the path $\gamma(t) = (t, 0)$, $0 \leq t \leq 1$, in the x-y plane. If $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, prove that

$$\lim_{n\to\infty} \int_{\gamma_n} f = \int_{\gamma} f,$$

where the integrals over γ and γ_n are path integrals.

3. Let $\omega = e^{\frac{2\pi i}{n}}$ with $n \in \mathbb{Z}$ and $n > 2$. Evaluate

$$1 + \omega^2 + \omega^4 + \cdots + \omega^{2(n-1)}.$$

4. Let $f(z)$ be defined of Im$z \geq 0$ such that f is analytic for Im$z > 0$, f is real if z is real, and f is continuous for all Im$z \geq 0$. Define

$$f(z) = \overline{f(\bar{z})}$$

for Im z<0.

a) Verify that f is differentiable for Im$z < 0$.

b) Verify that f is continuous for Im$z = 0$.

5. Evaluate $\int_0^\infty \frac{dx}{x^{1/5}(1+x)}$ using complex integration.