There are five (5) problems in this examination.

Problem 1

(a) (5 points) Let \(\{a, b\}\) be a basis for \(\mathbb{R}^2\) and \(A\) be a 2-by-2 matrix such that \(Aa = b\) and \(Ab = a\).

Find the eigenvalues and eigenvectors of \(A\) in terms of \(a\) and \(b\).

(b) (5 points) Show that if \(\{a, b\}\) is an orthonormal basis then \(\|A\|_2 = 1\).

(c) (10 points) Let \(\{c, d, e, f\}\) be a basis for \(\mathbb{R}^4\) and \(B\) be a 4-by-4 matrix such that \(Bc = d\), \(Bd = e\), \(Be = f\), and \(Bf = c\). Find the eigenvalues and the determinant of \(B\).

Solution

(a) \(a \pm b\) are eigenvectors with eigenvalues \(\pm 1\).

(b) Let \(V\) be the matrix with columns \(\{(a + b)/\sqrt{2}, (a - b)/\sqrt{2}\}\), which is also an orthonormal basis if \(\{a, b\}\) is. Then \(A^T A = V D^2 V^{-1}\) where \(D\) is diagonal with 1 and -1 on the main diagonal. Hence the singular values of \(A\), i.e. the square roots of the eigenvalues of \(A^T A\), are both 1.

(c) Let \(V\) be the matrix with columns \(\{c, d, e, f\}\). In the basis of \(V\), \(B\) becomes

\[
C = VBV^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]

and it has the same eigenvalues and determinant as \(B\). Using row operations we can compute \(\det(\lambda I - C)\) by reducing it to a diagonal matrix whose determinant is the product of the diagonal entries, \(\lambda^4 + 1\). The eigenvalues are thus the fourth roots of unity: \(\pm 1, \pm i\). The determinant is their product, -1.
Problem 2 

(a) (8 points) Consider the subspace $S$ of $\mathbb{R}^3$ given by $x + 2y + 3z = 0$ for $[x, y, z]^T \in \mathbb{R}^3$. Let $M$ be the matrix that reflects $\mathbb{R}^3$ through $S$. I.e. $Mu = u$ for $u \in S$ and $Mv = -v$ for $v \in S^\perp$, the orthogonal complement of $S$. Write $M$ explicitly, i.e. all of its entries.

(b) (4 points) Prove or disprove: A projection matrix (i.e. a matrix $P$ such that $P^2 = P$) may have an eigenvalue greater than 1.

(c) (8 points) Prove or disprove: A projection matrix may have a singular value greater than 1.

Solution

(a) Let $P$ be the orthogonal projection onto $S^\perp$. Then $M = I - 2P$. We have 

\[
P = \frac{1}{14} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & \end{bmatrix}; \quad M = \frac{1}{7} \begin{bmatrix} 6 & -2 & -3 \\ -2 & 3 & -6 \\ -3 & -6 & -2 \end{bmatrix}
\]

(b) This is false. If $\lambda$ is an eigenvalue, $\lambda v = Pv = P^2v = \lambda^2 v$, so $\lambda$ is 0 or 1.

(c) This is true if we have an oblique projection matrix $P$. The largest singular value of a projection matrix $P$ is $\|P\|_2 = \sup_v \|Pv\|_2/\|v\|_2$. Consider the rank-one projection $P = uw^T/w^Tu$ where $w^Tu \neq 0$. The range of $P$ is the span of $u$ and the null space is the orthogonal complement to the span of $w$. $\sup_v \|Pv\|_2/\|v\|_2 \geq \|Pw\|_2/\|w\|_2 = \|u\|_2 \|w\|_2/\|w^Tu\|$. So $P$ maps $w$ to a longer vector ($\|P\|_2 > 1$) as long as $w$ is not aligned with $u$.

Problem 3

Consider the differential equation

\[
d^2x \over dt^2 + b dx \over dt + cx + x^7 = 0.
\]  

(1)

with $x, t, b$, and $c$ real.

(a) (4 points) For $b > 0$, show that any solution to equation (1) remains bounded as $t \to +\infty$.

(b) (4 points) Let $x_1$ be a solution to equation (1) with $b = 0$ and $c > 0$. Let $x_1(0) = 0.1$ and $x_1'(0) = 0$. Let $x_1(t_0) = 0$ for some time $t_0$. What are the possible values of $x_1'(t_0)$?

(c) (4 points) Estimate $t_0$ from part b.

(d) (4 points) Prove that, with the initial conditions in part b, $y_1$ is the unique solution for some time interval.

(e) (4 points) Find the equilibria for $b = 0$ and $c < 0$. Using the concepts of kinetic and potential energy (without performing any detailed calculations), describe $x(t)$ for $t > 0$ given $x'(0) = 0$ and $x(0)$ that is slightly perturbed away from each equilibrium.

Solution

The equation describes a nonlinear spring with damping constant $b$ and stiffness $c$ (possibly negative) for the linear part of the spring force.
(a) We multiply by \( x' \) and obtain
\[
\frac{d}{dt} \left( \frac{1}{2} \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} cx^2 + \frac{1}{8} x^8 \right) = -b \left( \frac{dx}{dt} \right)^2 < 0.
\]
So \( \left( \frac{1}{2} \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} cx^2 + \frac{1}{8} x^8 \right) \) is bounded by its value at any finite time, and therefore \( \left( \frac{1}{2} cx^2 + \frac{1}{8} x^8 \right) \) is bounded by the same value. If \( |x| \) grows without bound, \( \left( \frac{1}{2} cx^2 + \frac{1}{8} x^8 \right) \) does also, whether \( c \) is positive or negative, because of the dominant \( x^8/8 \) term. This is a contradiction, and therefore \( |x| \) remains bounded.

(b) We have \( \left( \frac{1}{2} x^2_t + \frac{1}{2} cx^2 + \frac{1}{8} x^8 \right) = \text{constant} = \frac{5}{2} \times 10^{-2} + \frac{1}{8} \times 10^{-8} = \frac{1}{2} x_1'(t_0)^2 \),
so \( x_1'(t_0) = \pm \sqrt{c \times 10^{-2} + \frac{1}{8} \times 10^{-8}} \).

(c) We estimate \( t_0 \) by neglecting the nonlinear term. This is reasonable since \( |x_1| \leq 10^{-7} \). \( t_0 \) is shifted forward or backward from 0 by an integer multiple of a period plus or minus a quarter period (for the linear approximation). The period is \( 2\pi/\sqrt{c} \), so \( t_0 \approx 2\pi(n \pm 1/4)/\sqrt{c} \) for some integer \( n \).

(d) We write the equation as a first-order system \( x' = y; y' = -cx - x^7 \), in order to apply the nonlinear existence and uniqueness theorem. Since the right hand sides and their first partial derivatives with respect to \( x \) and \( y \) are continuous functions of \( x \) and \( y \) for all \( x \) and \( y \), there is some interval of time about the initial condition in which we have a unique solution.

(e) The equilibria have \(-cx - x^7 = 0 \) and \( y = x' = 0 \). There are three real solutions: \( x = 0, \pm(-c)^{1/6} \). \( x = 0 \) is a local maximum of potential energy, and \( x = \pm(-c)^{1/6} \) are local minima. For \( x(0) \) slightly perturbed away from \( x = \pm(-c)^{1/6} \), \( x(t) \) is a small periodic oscillation about these equilibria, which are stable. For \( x(0) \) slightly perturbed away from \( x = 0 \), \( x(t) \) oscillates periodically about whichever of \( \pm(-c)^{1/6} \) \( x(0) \) is closer to, such that the sum of kinetic and potential energy are conserved.

**Problem 4**

(a) (10 points) Find the solution of the initial value problem
\[
ty' + 2y = \frac{\cos t}{t}, \quad y(\pi/4) = 0.
\]

(b) (10 points) Find the form of the general solution to the ODE
\[
\frac{d^4y}{dt^4} + 2 \frac{d^3y}{dt^3} + 2 \frac{d^2y}{dt^2} = 5e^t + 2t^3e^{-t} + te^{-t} \sin t + e^{-t} \cos t.
\]

Write your answer in the form of a linear combination of functions of \( t \) with all of the coefficients left undetermined.

**Solution**

(a) Since this is a linear first order equation, we can use the integrating factor method. To find the integrating factor we first divide both sides by \( t \), and find the integrating factor as \( e^{\int 2dt/t} = t^2 \). We then multiply both sides by it to obtain
\[
t^2y' + 2ty = (t^2y)' = \cos t
\]
so
\[
y = \frac{\sin t}{t^2} - \frac{1}{\sqrt{2t^2}},
\]
with the constant in the second term fixed by the initial condition.
(b) We first find the solution to the homogeneous equation
\[
\frac{d^4y}{dt^4} + 2 \frac{d^3y}{dt^3} + 2 \frac{d^2y}{dt^2} = 0
\]
by inserting \( y = e^{rt} \). We have \( r = 0 \) with multiplicity 2 and \(-1 \pm i\). So the solution to the homogeneous equation is
\[
y = A + Bt + Ce^{-t} \sin t + De^{-t} \cos t.
\]
We then add terms for each type of exponential on the right hand side. If the exponential is multiplied by a polynomial in \( t \) we take an arbitrary polynomial in \( t \) of the same degree times that exponential. Then, if necessary, we multiply by the smallest power of \( t \) so that none of the terms overlap with those in the homogeneous solution. So for \( 5e^t \) we have \( 5e^t \); for \( 2t^3 e^{-t} \) we have \((Ft^3 + Gt^2 + Ht + I)e^{-t}\); for \( te^{-t} \sin t + e^{-t} \cos t \) (both exponentials of type \( e^{(-1 \pm i)t} \)) we have \( t(Jt + K)e^{-t} \sin t + t(Lt + M)e^{-t} \cos t \), the extra power of \( t \) to avoid overlap with the homogeneous solution. The full solution is
\[
y = A + Bt + Ce^{-t} \sin t + De^{-t} \cos t + 5e^t + (Ft^3 + Gt^2 + Ht + I)e^{-t}
+ t(Jt + K)e^{-t} \sin t + t(Lt + M)e^{-t} \cos t.
\]

**Problem 5**

Solve the PDE
\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) u = 0
\]
for \( u(r, \theta) \) in the angular sector \( \{ r > 0 ; -\pi/4 < \theta < \pi/4 \} \) with the boundary conditions:
\[
\frac{\partial u}{\partial \theta}(r, -\pi/4) = r^2, \quad u(r, \pi/4) = 1, \quad r > 0.
\]

**Solution**

We plug in a separation of variables solution \( u = R(r)Q(\theta) \) and obtain
\[
\frac{r^2 R''}{R} + \frac{r R'}{R} = -\frac{Q''}{Q} = \lambda^2.
\]
which has general solution \( R = r^{\pm \lambda} \) and \( Q = A \sin \lambda \theta + B \cos \lambda \theta \). We can match the boundary conditions by superposing solutions with \( \lambda = 0 \) and 2: \( u = A + Br^2 \sin 2\theta + Cr^2 \cos 2\theta \). The boundary condition at \(-\pi/4\) gives \( C = 1/2 \), and the boundary condition at \( \pi/4 \) gives \( A = 1, B = 0 \). The final answer is \( u = 1 + \frac{1}{2} r^2 \cos 2\theta \).