AIM Qualifying Review Exam in Differential Equations & Linear Algebra

September 2, 2017

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

(20 points) Consider the matrix $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 0 & d \\ 1 & 2 \end{pmatrix}$ and the vector $\mathbf{b} = \begin{pmatrix} e \\ 0 \\ 5 \end{pmatrix}$.

Find values for d and e such that the matrix equation $\mathbf{Ax} = \mathbf{b}$ has:

(a) Infinitely many solutions,

(b) A unique solution,

(c) No solutions.

Justify your answers.

Problem 1

Problem 1

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Problem 2

(20 points) Prove that there exists a real symmetric matrix $\mathbf{B}$ such that its second power

$$\mathbf{B}^2 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

by first showing that the given matrix is positive definite.

How many such $\mathbf{B}$ are there?

Problem 2

Problem 2

Problem 2

Problem 3

Consider the equation $y'' + y' + \sin y = 0$, describing the motion of a pendulum.

(a) (8 points) Determine all critical points and their stability.

(b) (8 points) For each type of critical point, sketch a phase portrait of the solutions in a neighborhood of the critical point in (y, y') space.

(c) (4 points) How does $y'^2/2 - \cos y$ evolve in time? Find a physical interpretation for this behavior, related to the pendulum.

Problem 3

Problem 3

Problem 3

Problem 4

(a) (10 points) Solve the initial value problem

$$y'' - 2y' + y = te^t,\ y(0) = 1,\ y'(0) = 1.$$

(b) (10 points) Consider the eigenvalue problem $Lu = \lambda u$ with L an operator acting on functions u according to the formula

$$Lu = u'' + u,$$

where u satisfy the boundary conditions

$$u(0) - u'(1) = 0,\ u'(0) - u(1) = 0.$$

Is L a self-adjoint operator on this space of functions?

Problem 4

Problem 4

Problem 4

Problem 5

(a) (8 points) Find the solution to

$$\partial_{tt}u(x,t) = \partial_{xx}u(x,t),\ -\infty < x < \infty,\ t > 0$$
$$u(x,0) = x^2(1-x)^2,\ 0 \le x \le 1,\ u(x,0) = 0 \text{ elsewhere, and } \partial_t u(x,0) \equiv 0.$$

Sketch the solution at $t = 2$.

(b) (12 points) Solve the following equation with given initial and boundary conditions

$$\partial_t w(x,t) - \partial_{xx}w(x,t) = 0,\ 0 \le x \le 1,\ t > 0$$
$$w(x,0) = 0,\ 0 \le x \le 1$$
$$w(0,t) = 0,\ w(1,t) = 1,\ t > 0.$$

Sketch the solution at positive times $t = t_1 \ll 1$ and $t = t_2 \gg 1$.

Problem 5

Problem 5

Problem 5