AIM Qualifying Review Exam in Differential Equations & Linear Algebra

January 2024

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

Let $\mathbf{M}_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $\mathbf{M}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$, $\mathbf{M}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{pmatrix}$, etc.,
so for general n, $\mathbf{M}_n$ is the n-by-n matrix with all entries equal to 1 on the main diagonal, all entries equal to n on the first subdiagonal, and zeros elsewhere.

(a) Find the determinant of $\mathbf{M}_n$ and justify your answer.

(b) Show that all eigenvectors of $\mathbf{M}_n$ lie in the span of a single vector and give its entries.

(c) Find all the entries of the inverse of $\mathbf{M}_n$.

Solution

(a) For a triangular matrix, the eigenvalues are the diagonal entries and the determinant is their product, 1 in this case. There are many other ways to get the same answer. For example, one could note that the determinant of the identity matrix is 1, and $\mathbf{M}_n$ can be reduced to the identity by subtracting n times each row from the row below, which does not change the determinant.

(b) Since all the eigenvalues are 1, all the eigenvectors of $\mathbf{M}_n$ are in the kernel of $\mathbf{I} - \mathbf{M}_n$, which has zeros on the main diagonal and $-n$s on the first subdiagonal. The range of this matrix is the column space, the span of $-n\mathbf{e}_2, \ldots, -n\mathbf{e}_n$, so its rank is $n-1$ and hence its nullity is 1 by the rank-plus-nullity theorem. Hence one vector spans the eigenspace, and we can take it to be $\mathbf{e}_n$, the vector with last entry 1 and the rest zero.

(c) We use the Gauss-Jordan elimination method to compute the inverse. I.e. we add the n-by-n identity matrix to the right of $\mathbf{M}_n$, and perform row operations on the resulting n-by-$2n$ matrix so the left half transforms from $\mathbf{M}_n$ to the identity, while the right half transforms from the identity to $\mathbf{M}_n^{-1}$. To do this, we subtract n times the first row from the second, then n times the second row from the third, ..., until, at the last step, we subtract n times the $(n-1)$st row from the nth. The result for the right side, $\mathbf{M}_n^{-1}$, has ones on the main diagonal, $-n$ on the first subdiagonal, $(-n)^2$ on the second subdiagonal, ..., $(-n)^{n-1}$ on the $(n-1)$ subdiagonal, the single lower left corner entry.

Problem 2

(a) Find the line $C + Dt$ that gives a least-squares fit to $b = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$.

(b) Find the vector $\mathbf{c}_1$ that is the orthogonal projection of $\mathbf{a} = (1, 2)$ onto the line spanned by $\mathbf{b}_1 = (1, 0)$. Find the vector $\mathbf{c}_2$ that is the orthogonal projection of $\mathbf{a} = (1, 2)$ onto the line spanned by $\mathbf{b}_2 = (1, 1)$. Show that $\mathbf{a} \neq \mathbf{c}_1 + \mathbf{c}_2$.

(c) Show rigorously that for any nonzero vectors $\mathbf{B}_1$ and $\mathbf{B}_2 \in \mathbb{R}^2$, the orthogonal projections of $\mathbf{A}$ onto $\mathbf{B}_1$ and $\mathbf{B}_2$ add up to $\mathbf{A}$ for all $\mathbf{A} \in \mathbb{R}^2$ if and only if $\mathbf{B}_1$ and $\mathbf{B}_2$ are orthogonal to each other.

Solution

(a) We minimize $\|\mathbf{Ax} - \mathbf{b}\|_2$ where $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} C \\ D \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$. The solution is $\mathbf{x} = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*\mathbf{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(b) $\mathbf{c}_1 = \mathbf{b}_1(\mathbf{a}^*\mathbf{b}_1)/(\mathbf{b}_1^*\mathbf{b}_1) = (1, 0)$ and $\mathbf{c}_2 = \mathbf{b}_2(\mathbf{a}^*\mathbf{b}_2)/(\mathbf{b}_2^*\mathbf{b}_2) = (3/2, 3/2)$, so $\mathbf{c}_1 + \mathbf{c}_2 = (5/2, 3/2)$.

(c) First, assume $\mathbf{A} = \mathbf{B}_1(\mathbf{A}^*\mathbf{B}_1)/(\mathbf{B}_1^*\mathbf{B}_1) + \mathbf{B}_2(\mathbf{A}^*\mathbf{B}_2)/(\mathbf{B}_2^*\mathbf{B}_2)$ for all $\mathbf{A}$. Take the inner product of both sides with $\mathbf{B}_1$ and get $\mathbf{B}_1^*\mathbf{A} = \mathbf{A}^*\mathbf{B}_1 + (\mathbf{B}_1^*\mathbf{B}_2)(\mathbf{A}^*\mathbf{B}_2)/(\mathbf{B}_2^*\mathbf{B}_2)$. This holds for all $\mathbf{A}$ iff $\mathbf{B}_1^*\mathbf{B}_2 = 0$. Now assume $\mathbf{B}_1^*\mathbf{B}_2 = 0$. Let $\mathbf{F} = \mathbf{A} - \mathbf{B}_1(\mathbf{A}^*\mathbf{B}_1)/(\mathbf{B}_1^*\mathbf{B}_1) - \mathbf{B}_2(\mathbf{A}^*\mathbf{B}_2)/(\mathbf{B}_2^*\mathbf{B}_2)$. Show that $\mathbf{F}^*\mathbf{B}_1 = \mathbf{F}^*\mathbf{B}_2 = 0$. Since $\mathbf{B}_1$ and $\mathbf{B}_2$ are a basis, $\mathbf{F}$ is orthogonal to every vector in $\mathbb{R}^2$, so $\mathbf{F}$ must be the zero vector for every $\mathbf{A} \in \mathbb{R}^2$, and thus $\mathbf{A}$ is the sum of the two projections.

Problem 3

(a) Find the general solution of the following differential equation:

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^t$$

(b) Solve $ty' + (1 + t)y = t$ with the condition $y(1) = 1$.

Solution

(a) First we find the eigenvalues and eigenvectors of the matrix to obtain the solution to the homogeneous equation as $\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$. Then we try a particular solution in the form $\mathbf{A}e^t$. The full solution is $\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} + \begin{bmatrix} 1/4 \\ -2 \end{bmatrix} e^t$.

(b) This equation is of the form $y' + p(t)y = q(t)$, so it can be solved by the integrating factor method. The integrating factor is $e^{\int p(t)dt} = te^t$, and the equation can be written $(te^t y)' = te^t$. Integrating and applying the boundary condition we obtain $y = 1 - 1/t + e^{1-t}/t$.

Problem 4

(a) Let $\Phi(t)$ be a fundamental matrix for the following system of equations (i.e. $\Phi(t)$ is a matrix-valued function whose columns are linearly independent solutions of the system):

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \mathbf{x}$$

Find the fundamental matrix $\Phi(t)$ that obeys the initial condition $\Phi(0) = \mathbf{I}$. Your solution should be a matrix with each entry given explicitly as a function of t.

(b) Find the general solution to the ODE

$$t^3 \frac{d^3y}{dt^3} + t^2 \frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2y = 2t^4 \, , \; t > 0.$$

Solution

(a) Finding the eigenvalues and eigenvectors, we get the general solution as $\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{-2it}$. The first column of $\Phi(t)$ is the linear combination that gives $\mathbf{e}_1$ at $t = 0$, i.e. $c_1 = c_2 = 1/2$. The second column of $\Phi(t)$ is the linear combination that gives $\mathbf{e}_2$ at $t = 0$, i.e. $c_1 = 1/4i, c_2 = -1/4i$. Thus $\Phi(t) = \begin{bmatrix} \cos 2t & \frac{1}{2}\sin 2t \\ -2\sin 2t & \cos 2t \end{bmatrix}$.

(b) We first find the solution to the homogeneous equation

$$t^3 \frac{d^3y}{dt^3} + t^2 \frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2y = 0$$

by inserting $y = t^r$. We have a cubic equation in r with solutions -1, 1, and 2. So the solution to the homogeneous equation is

$$y = A/t + Bt + Ct^2.$$

We try a particular solution in the form Dt^4 and find $D = 1/15$. The full solution is

$$y = A/t + Bt + Ct^2 + t^4/15.$$

Problem 5

(a) Write the general solution of the PDE

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

for $u(x, t)$ with the boundary conditions:

$$u(0, t) = 0, \quad \frac{\partial u}{\partial x}(1, t) = 1.$$

(b) Give the particular solution to part (a) that also satisfies the initial condition

$$u(x, 0) = x + \sin(5\pi x/2).$$

Solution

(a) We write $u = x + v$, where x is the steady-state solution and v has homogeneous boundary conditions. We plug in a separation of variables solution $v = T(t)X(x)$ and obtain

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda^2 < 0.$$

We have $X(x) = A\sin(\lambda x) + B\cos(\lambda x)$. Nontrivial solutions that satisfy the homogeneous boundary conditions are given by $X(x) = a_1 \sin(\pi x/2), a_2 \sin(3\pi x/2), a_3 \sin(5\pi x/2), \ldots$. If instead we chose a separation constant $\lambda^2 > 0$, we would have $X(x) = A\sinh(\lambda x) + B\cosh(\lambda x)$, and we would have $A = B = 0$ for all real λ. Meanwhile, we have $T(t) = Ce^{-\lambda^2 t}$ for $\lambda = \pi/2, 3\pi/2, \ldots$. The general solution is a superposition of these solutions:
$v(x, t) = \sum_{k=1}^{\infty} a_k e^{-((2k-1)\pi/2)^2 t)} \sin((2k-1)\pi x/2)$.

(b) We can match the initial condition with $a_3 = 1$ and the remaining a_k zero:
$u(x, t) = x + e^{-(5\pi/2)^2 t} \sin(5\pi x/2)$.