AIM Qualifying Review Exam in Differential Equations & Linear Algebra

January 2024

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

Let $\mathbf{M}_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $\mathbf{M}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$, $\mathbf{M}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{pmatrix}$, etc., so for general n, $\mathbf{M}_n$ is the n-by-n matrix with all entries equal to 1 on the main diagonal, all entries equal to n on the first subdiagonal, and zeros elsewhere.

(a) Find the determinant of $\mathbf{M}_n$ and justify your answer.

(b) Show that all eigenvectors of $\mathbf{M}_n$ lie in the span of a single vector and give its entries.

(c) Find all the entries of the inverse of $\mathbf{M}_n$.

Problem 1

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Problem 2

(a) Find the line $C + Dt$ that gives a least-squares fit to $b = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$.

(b) Find the vector $\mathbf{c}_1$ that is the orthogonal projection of $\mathbf{a} = (1, 2)$ onto the line spanned by $\mathbf{b}_1 = (1, 0)$. Find the vector $\mathbf{c}_2$ that is the orthogonal projection of $\mathbf{a} = (1, 2)$ onto the line spanned by $\mathbf{b}_2 = (1, 1)$. Show that $\mathbf{a} \neq \mathbf{c}_1 + \mathbf{c}_2$.

(c) Show rigorously that for any nonzero vectors $\mathbf{B}_1$ and $\mathbf{B}_2 \in \mathbb{R}^2$, the orthogonal projections of $\mathbf{A}$ onto $\mathbf{B}_1$ and $\mathbf{B}_2$ add up to $\mathbf{A}$ for all $\mathbf{A} \in \mathbb{R}^2$ if and only if $\mathbf{B}_1$ and $\mathbf{B}_2$ are orthogonal to each other.

Problem 2

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Problem 3

(a) Find the general solution of the following differential equation:

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^t$$

(b) Solve $ty' + (1+t)y = t$ with the condition $y(1) = 1$.

Problem 3

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Problem 4

(a) Let $\Phi(t)$ be a fundamental matrix for the following system of equations (i.e. $\Phi(t)$ is a matrix-valued function whose columns are linearly independent solutions of the system):

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \mathbf{x}$$

Find the fundamental matrix $\Phi(t)$ that obeys the initial condition $\Phi(0) = \mathbf{I}$. Your solution should be a matrix with each entry given explicitly as a function of t.

(b) Find the general solution to the ODE

$$t^3 \frac{d^3y}{dt^3} + t^2 \frac{d^2y}{dt^2} - 2t \frac{dy}{dt} + 2y = 2t^4 \,, \; t > 0.$$

Problem 4

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Problem 5

(a) Write the general solution of the PDE

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

for $u(x,t)$ with the boundary conditions:

$$u(0,t) = 0, \ \ \frac{\partial u}{\partial x}(1,t) = 1.$$

(b) Give the particular solution to part (a) that also satisfies the initial condition

$$u(x,0) = x + \sin(5\pi x/2).$$

Problem 5

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