AIM Qualifying Review Exam: Probability and Discrete Mathematics

January 8, 2024

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Let M, B, K denote the the outputs of rolling a maize, blue, and khaki die, respectively. (A die has six sides, labeled $1, 2, \ldots, 6$, and each side comes up with equal probability $1/6$ when the die is rolled.)

(a) What is the probability that no two dice land on the same number?

(b) Given that no two of the dice land on the same number, what is the conditional probability that $M < B < K$?

(c) What is $P(M < B < K)$?

Problem 1

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Problem 2 Let Y be a continuous random variable without point masses and with density f_Y (so $\int_{-\infty}^{\infty} f_Y(y)dy = 1$). Show that the expectation $\mathbb{E}[Y]$ is

$$\mathbb{E}[Y] = \int_0^{\infty} P\{Y > y\}dy - \int_0^{\infty} P\{Y < -y\}dy$$

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Problem 2

Problem 2

Problem 2

Problem 3

The n'th triangular number, A_n, is defined as $A_n = 1 + 2 + 3 + \cdots + n$. The n'th tetrahedral number, V_n, is defined as $V_n = A_1 + A_2 + \cdots + A_n$. (Since triangle and tetrahedron start with the same letter, use A as in "area" for triangles and V as in "volume" for tetrahedra.)

(a) Find a formula (without $\cdots$) for A_n.

(b) Show by induction that $V_n = \binom{n+2}{3}$, the binomial symbol $\binom{n+2}{3} = \frac{(n+2)(n+1)n}{1 \cdot 2 \cdot 3}$.

(c) Show by induction that $V_{2n} = 2^2 + 4^2 + 6^2 + \cdots + (2n)^2$. (Note even index in V_{2n}.)

Best wishes for $2024 = V_{22} = 1 + (1+2) + (1+2+3) + \cdots + (1+2+\cdots+22) = 2^2 + 4^2 + \cdots + 22^2 = \binom{22+2}{3}$.

Problem 3

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Problem 4

(a) Show that if 51 distinct numbers are chosen from $1, 2, \cdots, 100$, then there is a pair that differs by 1.

(b) Show that if 5 points are chosen in the unit square, some pair of points is at distance at most $\frac{\sqrt{2}}{2}$.

Problem 4

Problem 4

Problem 4

Problem 5 The Karatsuba algorithm multiplies two n-term polynomials, p and q, by divide-and-conquer,

more efficiently than the gradeschool algorithm, that looks at all n^2 pairs of a term in p and a term in q. As a base case, $(a_0 + a_1 x)(b_0 + b_1 x)$ is done by computing $a_0 b_0, (a_0 + a_1)(b_0 + b_1)$, and $a_1 b_1$. To multiply $(a_0 + a_1 x + a_2 x^2 + a_3 x^3)(b_0 + b_1 x + b_2 x^2 + b_3 x^3)$, put

$$\begin{aligned} u &= x^2 \\ A_0 &= a_0 + a_1 x \\ A_1 &= a_2 + a_3 x \\ B_0 &= b_0 + b_1 x \\ B_1 &= b_2 + b_3 x, \end{aligned}$$

then do $(A_0 + A_1 u)(B_0 + B_1 u)$, using the base case algorithm on capital letters and on recursive calls: for example, $A_1 B_1$ is unwound to $(a_2 + a_3 x)(b_2 + b_3 x)$. The general case for larger n is similar.

(a) This algorithm can be described as taking a problem of size n and reducing it to s problems of size at most n/t. What are s and t?

(b) Suppose the original coefficients $a_0, a_1, \ldots$ are reals. How many multiplications of real numbers are incurred by the algorithm when $n = 4$? (Do not simplify in case of zero coefficients, etc. And do **not** count $A_1 B_1$ as a multiplication, since it is a virtual multiplication of polynomials, not a multiplication of real numbers.)

(c) How many multiplications of real numbers are incurred for general n? Answer in the form $O(n^r)$, i.e., give the exponent r, in worst case, without need to calculate a constant factor c in cn^r.

Problem 5

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