AIM Preliminary Exam: Probability and Discrete Mathematics

August 20, 2020

There are five (5) problems in this examination.

You may write your answers on these pages or separate paper. Be sure to put your secret number in the top right corner of each page. When you are finished, write the number of each page of out the total number of pages i.e. 1/6, 2/6 etc.

Problem 1

Each of 500 soldiers independently has a certain disease with probability 10^{-3}. The disease will show up in a blood test. Samples from all 500 are pooled and a single test is performed.

(a) What is the probability that the test will be positive (that is, that at least one person has the disease)? Give an exact expression and an approximate value to within 20%.

(b) Suppose now that the test indeed yields a positive result. What is the conditional probability that exactly one person has the disease?

Problem 2 Suppose the number of miles that a car can run before its battery wears out is exponentially distributed with an average of 10,000 miles. That is, the density is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0; \\ 0, & x < 0, \end{cases}$$

where $\lambda = 10^{-4}$.

(a) If a person desires to take a 5000-mile trip, what is the probability that they will be able to complete the trip without having to replace the battery? Write an exact expression and approximate the value.

(b) What can be said if the battery-life distribution is **not** exponential?

Problem 3

What is the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 18$$

that satisfy

$$\begin{array}{rcll} 1 & \leq & x_1 & \leq 5 \\ -2 & \leq & x_2 & \leq 4 \\ 0 & \leq & x_3 & \leq 5 \\ 3 & \leq & x_4 & \leq 9? \end{array}$$

You may leave binomial symbols unevaluated. Give a representative amount of work—enough to demonstrate that you know how to complete the problem—but you may omit repetitive work for similar cases.

Problem 4

In the Tower of Hanoi problem, there are three pegs A, B, and C and discs of diameters $1, 2, 3, \ldots, n$. The discs have holes at their centers and are stacked on pegs, with the constraint that no disc is atop a disc of smaller diameter. Initially all discs are on peg A and the goal is to move all discs, one at a time, so the stack ends on peg B. A recursive algorithm first moves the top $n-1$ discs to peg C, moves disc n to B, then moves the $n-1$ discs from C to B.

(a) How many moves are required, as a function of n? Carefully give an inductive proof.

(b) What is $\sum_{i=0}^{n} i^3$? Give an inductive proof.

Problem 5

The margin alignment problem is to take text with arbitrary margins, like

```
Call me
Ishmael.  Some years ago, never mind how
long precisely, having little
or no money in my purse and nothing particular to interest me on
shore, I thought I would sail about
a little and see the watery part of the world.
```

and reformat it to make the right margin as even as possible, like

```
Call me Ishmael. Some years ago, never|
mind how long precisely, having little|
or no money in my purse and nothing   |
particular to interest me on shore, I |
thought I would sail about a little   |
and see the watery part of the world. |
```

Suppose there is fixed, given maximum line length of L and our only freedom is to choose to terminate a line and to leave more spaces at the right margin. (We are using a fixed-width font and there is always exactly one space between words. Punctuation counts as ordinary letters. Assume no word has more than L characters.)

If a line has $L - s$ characters followed by s spaces, then s is the **slack** of that line.

Let n be the number of words. Give an algorithm with run time polynomial in $(n + L)$ to partition the words into lines to minimize the sum of the squares of the slacks. (In the above, the sum of squares is $0 + 0 + 9 + 1 + 9 + 1 = 20$.) Analyze the runtime in terms of n and L using big-O notation.