AIM Preliminary Exam: Advanced Calculus and Complex Variables

September 2, 2017

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1 Consider the integral I_p given by

$$I_p := \int_{-\infty}^{+\infty} \frac{e^{px}\,dx}{1+e^x}.$$

(a) Find the largest set $D \subset \mathbb{C}$ such that the integral converges absolutely for $p \in D$.

(b) Using contour integration in the complex x-plane, evaluate I_p assuming $p \in D$.

Problem 1

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Problem 2 Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) := x^2/(1+x^2)^n$.

(a) Determine all values of $x \in \mathbb{R}$ for which $\sum_{n=0}^{\infty} f_n(x)$ converges, and find the sum $S(x)$ in closed form.

(b) Determine all intervals (finite or infinite) on which the convergence of the series is uniform.

Problem 2

Problem 2

Problem 2

Problem 3

(a) Let $a \in \mathbb{C}$ be a constant and consider the polynomial

$$P(z) = z^{10} + a(z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1).$$

Given a radius $\rho > 0$, use complex analysis to find an upper bound on $|a|$ of the form $|a| < \delta(\rho)$, $\delta(\rho) > 0$, sufficient to guarantee that P has all ten of its zeros in the disk $|z| < \rho$.

(b) Let coefficients c_n be defined by

$$c_n := \frac{1}{2\pi i} \oint_C \frac{dz}{z^{n+1} \cosh(z)}, \quad n = 0, 1, 2, 3, \ldots$$

where C is a small circle going around the origin in the positive direction. What is the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n z^n$?

Problem 3

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Problem 3

Problem 4 For $n \geq 1$, define continuous functions $f_n : [0, \infty) \to \mathbb{R}$ by

$$f_n(x) := \begin{cases} e^{-x}, & 0 \leq x \leq n \\ e^{-2n}(e^n + n - x), & n \leq x \leq n + e^n \\ 0, & x \geq n + e^n. \end{cases}$$

(a) Find the pointwise limit $f(x) := \lim_{n\to\infty} f_n(x)$ for $x \geq 0$.

(b) Is the convergence uniform on $[0, \infty)$? Prove or disprove.

(c) Compute

$$\int_0^\infty f(x)\, dx \quad \text{and} \quad \lim_{n\to\infty} \int_0^\infty f_n(x)\, dx$$

and comment in light of part (b).

Problem 4

Problem 4

Problem 4

Problem 5

(a) Let S be the closed surface whose bottom face B is the unit disc in the (x, y)-plane and whose upper part U is the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$. Use the divergence theorem to calculate the flux of $\vec{F}(x, y, z) = (x, y, z)^\top$ out of S through the upper part U.

(b) An (x, z)-*cylinder* is a surface in $\mathbb{R}^3$ whose points satisfy an equation of the form $f(x, z) = 0$ (no y-dependence); hence its cross-section through any plane $y = c$ perpendicular to the y-axis is always the same curve in the (x, z)-plane. Consider the vector field $\vec{G}(x, y, z) := (z^2, y^2, xz)^\top$ defined on $\mathbb{R}^3$. Show that whenever C is a simple closed curve lying on an (x, z)-cylinder, $\oint_C \vec{G} \cdot d\vec{r} = 0$.

Problem 5

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