Advanced Calculus and Complex Variables (solution hints)
January 2022.
For full credit, support your answers with appropriate explanations.
There are five problems, each worth 20 points.

1. Suppose $f : [0,1] \to \mathbb{R}$ is a function that is strictly increasing and continuous with $f(0) < f(1)$. Suppose that

$$f(0) = y_0 < y_1 < \cdots < y_n = f(1)$$

is a division of the interval $[f(0), f(1)]$.

(a) (12 points) For $I_j = [y_j, y_{j+1}]$, $j = 0, \ldots, n-1$, prove that $f^{-1}(I_j)$ is a closed interval. Denote the length of $f^{-1}(I_j)$ by μ_j and let

$$S = \sum_{j=0}^{n-1} y_j \mu_j.$$

If $\delta = \max_{j=0,\ldots,n-1} |y_j - y_{j+1}|$, does S converge as $\delta \to 0$ and if so to what quantity? Justify your answer.

(b) (8 points) Suppose now that $f(0) < f(1)$ and that f is increasing but not strictly increasing. Does S converge as $\delta \to 0$ and if so to what quantity?

Solution (a) S converges to $\int_0^1 f\,dx$. For each y_j there is a unique point x_j such that $f(x_j) = y_j$. We have $x_0 = 0$ and $x_n = 1$ and the intervals $f^{-1}(I_j)$ overlap only at their endpoints. Then

$$y_j \mu_j < \int_{x_j}^{x_{j+1}} f\,dx < y_{j+1}\mu_j$$

and

$$0 < \int_0^1 f dx - \sum_{j=0}^{n-1} y_j \mu_j < \sum (y_{j+1} - y_j)\mu_j \leq \delta.$$

Thus S must converge to $\int_0^1 f\,dx$.
(b) If f is increasing but not necessarily strictly $f^{-1}[y_{j-1}, y_j] \cap f^{-1}[y_j, y_{j+1}]$ may be an interval of length strictly greater than zero. It is easy to find examples for which S does not converge. For instance, consider

$$f(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \leq x \leq \frac{1}{2} \\ x - \frac{1}{4} & \frac{1}{2} \leq x \leq 1. \end{cases}$$

If we consider subdivisions in which $y_j = \frac{1}{4}$ for some j, the interval $x \in [\frac{1}{4}, \frac{1}{2}]$ is counted twice being a subset of both $f^{-1}[y_{j-1}, y_j]$ and $f^{-1}[y_j, y_{j+1}]$. Thus the limit as $\delta \to 0$ will be $\int_0^1 f + \frac{1}{16}$. If we consider subdivisions for which $y_j \neq \frac{1}{4}$ for any j, then the limit as $\delta \to 0$ will be $\int_0^1 f$.

2. (20 points) Suppose $f : [0,1] \to \mathbb{R}$ is an increasing function but not necessarily continuous. Let S denote the set of points $x \in [0,1]$ at which f is discontinuous. Prove that S is a countable set.

Solution Define

$$\delta(x) = \lim_{\epsilon \to 0^+} f(x+\epsilon) - \lim_{\epsilon \to 0^+} f(x-\epsilon).$$

It follows that $f(x)$ is continuous at x if and only if $\delta(x) = 0$. Suppose $\delta(x) \geq \frac{1}{k}$ at $x_1 < \ldots < x_n$. We must then have $\frac{n}{k} \leq f(1) - f(0)$ which implies that n must be finite. Thus S_k, the set of x with $\delta(x) \geq \frac{1}{k}$, is finite for $k = 1, 2, 3, \ldots$. The set of discontinuities must be contained in $\cup S_k$, which is a countable set.

3. Let $f(z) = \arctan z$ with $f(0) = 0$.

   (a) (5 points) Obtain the Taylor series of $f(z)$ in the form

   $$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

   and determine its radius of convergence.

   (b) (15 points) Suppose $f(z)$ is expanded in a Taylor series of the form

   $$f(z) = a_0 + a_1(z-1) + a_2(z-1)^2 + \cdots$$

   with $a_0 = f(1) = \frac{\pi}{4}$. What is its radius of convergence?

Solution $\arctan z$ has singularities at $z = \pm i$, which may be easily verified by taking its derivative. In (a) $R = 1$ and in (b) $R = \sqrt{2}$. The radius of convergence is the distance from the center of the power series to the closest singularity. In (a), R may be obtained by calculating the Taylor series explicitly.

4. (20 points) Let $f(z)$ be a complex valued function that is continuous for $|z| \leq 1$ and analytic for $|z| < 1$. Assume that $|f(z)| = 1$ if $|z| = 1$. Suppose the definition of f is extended to $|z| > 1$ using

$$f(z) = \frac{1}{\bar{f}\left(\frac{1}{\bar{z}}\right)},$$

where $\bar{z}$ is the complex conjugate.

   (a) Show that $f(z)$ has a complex derivative for $|z| > 1$ and calculate this derivative in terms of f' evaluated for $|z| < 1$.

   (b) Show that $f(z)$ is continuous for $|z| = 1$.

Solution We may verify via a calculation that

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \frac{\bar{f}'\left(\frac{1}{\bar{z}}\right)}{\left(\bar{f}\left(\frac{1}{\bar{z}}\right)\right)^2 z^2}$$

for $|z| > 1$. The first step in this calculation is

$$\bar{f}\left(\frac{1}{\bar{z}+\bar{h}}\right) = \bar{f}\left(\frac{1}{\bar{z}}\right) + \bar{f}'\left(\frac{1}{\bar{z}}\right)\left(-\frac{h}{z^2}\right) + \cdots$$

which is obtained by conjugating the Taylor series

$$\begin{aligned} f\left(\frac{1}{\bar{z}+\bar{h}}\right) &= f\left(\frac{1}{\bar{z}} - \frac{\bar{h}}{\bar{z}^2} + \frac{\bar{h}^2}{\bar{z}^3} - \cdots\right) \\ &= f\left(\frac{1}{\bar{z}}\right) + f'\left(\frac{1}{\bar{z}}\right)\left(-\frac{\bar{h}}{\bar{z}^2}\right) + \cdots \end{aligned}$$

We then use

$$\begin{aligned} \frac{f(z+h) - f(z)}{h} &= \frac{\frac{1}{\bar{f}\left(\frac{1}{\bar{z}+\bar{h}}\right)} - \frac{1}{\bar{f}\left(\frac{1}{\bar{z}}\right)}}{h} \\ &= \frac{1}{\bar{f}\left(\frac{1}{\bar{z}+\bar{h}}\right)\bar{f}\left(\frac{1}{\bar{z}}\right)} \frac{\bar{f}\left(\frac{1}{\bar{z}}\right) - \bar{f}\left(\frac{1}{\bar{z}+\bar{h}}\right)}{h}. \end{aligned}$$

The answer is obtained by substituting for the numerator and taking the limit $h \to 0$. For $|z| = 1$, the extended f is continuous because $z \to \frac{1}{\bar{z}}$ maps every point on the unit circle to itself.

5. (20 points) Evaluate

$$\int_0^\infty \frac{dx}{x^{\frac{1}{4}}(1+x)}.$$

Solution Let $I = \int_0^\infty \frac{dx}{x^{\frac{1}{4}}(1+x)}$. Define $z^{\frac{1}{4}} = re^{\frac{i\theta}{4}}$ with $\theta = \arg(z) \in (0, 2\pi)$. Thus the branch cut is the positive real line. Let γ be the contour that goes from $z = 0$ to $z = R$ slightly above the branch cut, traverses $|z| = R$ in counterclockwise sense, and then goes from $z = R$ to $z = 0$ slightly below the branch cut. Then we have

$$(1+i)I = \lim_{R\to\infty} \int_\gamma \frac{dz}{z^{\frac{1}{4}}(1+z)} = 2\pi i \text{Res}\left(\frac{1}{z^{\frac{1}{4}}(1+z)}; z = -1\right) = \frac{\sqrt{2}2\pi i}{(1+i)}.$$

Thus $I = \sqrt{2}\pi$.