AIM Qualifying Review Exam: Probability and Discrete Mathematics

August 19, 2022

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

In a game, our goal is to determine exactly an unknown integer, $1, 2, \ldots, 1024 = 2^{10}$, chosen uniformly at random, by asking yes/no questions.

(a) What is the expected number of questions needed if we ask: Is it 1? Is it 2? (asking integers sequentially).

(b) What is the expected number of questions if we eliminate exactly half the possibilities with each question?

Show work. (Hint: Consider cases smaller than 1024 to check for off-by-1 errors.)

Solution

Ross 4.26, p. 175.

Put $N = 1024$. In the first case, if the random number is $i < N$, it takes i questions.

Note that, with a careful reading, we never have to ask the N'th question, because we know the answer after $N - 1$ nos.

So the expected number is $\frac{1}{N}\sum_{i=1}^{N-1} i + \frac{N-1}{N} = \frac{1}{N}\binom{N+1}{2} - \frac{1}{N} = \frac{N+1}{2} - \frac{1}{N} = \frac{N^2+N-2}{2N}$.

An alternative, deprecated, reading requires the last question (and gives a prettier answer of) $\frac{N+1}{2}$.

In the second case, we never learn the number exactly until the last question, the tenth, at which point we always learn the number exactly. So the expected number of questions is 10.

Mathematical concepts: discrete random variables, expectation

Problem 2

Alice and Bob agree to meet at M36 coffee shop, but each arrives independently at random between noon and 1pm. What is the probability that the first to arrive has to wait more than 10 minutes?

Solution

Ross, pp. 243, Example 2c.

Let X and Y be the arrival times. By symmetry, we want $2\Pr(X+10<Y)$. This is twice the relative area of a triangle within the 60-by-60 square bounded by $X=0, Y=60, Y=X+10$. This is a 45-45-90 triangle with short side 50 minutes, so twice the relative area is $\left(\frac{50}{60}\right)^2 = \frac{25}{36}$.

Alternatively, the area can be found using an integral.

Mathematical concepts: Independent random variables

Problem 3

Find the number of integers between 1 and 24,000, inclusive, that are not divisible by 4, 5, or 6.

Solution

Brualdi 6,7.1, page 198.

Note that 4 and 6 share a factor, and note that $4 \cdot 5 \cdot 6$ divides 24,000.

Let A_4, A_5, A_6 be the sets of multiples of 4, 5, 6, respectively. Then we want:

$$24000 - (|A_4| + |A_5| + |A_6|) + (|A_4 \cap A_5| + |A_4 \cap A_6| + |A_5 \cap A_6|) - (|A_4 \cap A_5 \cap A_6|).$$

This is $24000\left(1 - \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right) + \left(\frac{1}{20} + \frac{1}{12} + \frac{1}{30}\right) - \frac{1}{60}\right)$.

Here we used that $|A_4 \cap A_6| = 24000 \cdot \frac{1}{12}$, not $24000 \cdot \frac{1}{4 \cdot 6}$, and similarly for $A_4 \cap A_5 \cap A_6$.

Mathematical concepts: Inclusion-exclusion principle

Problem 4

Show that, for any sequence of n^2+1 distinct real numbers $a_1, a_2, \ldots, a_{n^2+1}$, either there is an increasing subsequence of length $n+1$ or a decreasing subsequence of length $n+1$.

Note: A subsequence is gotten by deleting zero or more elements from the original sequence, while retaining the order of the remaining terms. "Increasing" or "decreasing" refers to values. For example, given sequence $(5,1,2,9,8)$, the sequence $(5,2)$ is a subsequence, gotten by striking 1, 9, 8 and leaving 5 and 2 in their original order with 5 preceding 2. It is decreasing, since $2<5$.

Solution

Brualdi, p.76, Application 9

Let m_k denote the length of the longest increasing subsequence that begins at k. Assume that, for all k, we have $m_k \le n$. Since there are n^2+1 such m_k's, the strong pigeonhole principle guarantees at least $n+1$ equal m_k's, say $m_{k_1} = m_{k_2} = \cdots m_{k_{n+1}}$, where we take $k_1 < k_2 < \cdots < k_{n+1}$.

Because each m_{k_i} is defined in terms of an increasing sequence of maximal length, we must have $a_{k_1} > a_{k_2} > \cdots > a_{k_{n+1}}$.

Mathematical concepts: (Strong) pigeonhole principle

Problem 5

We are given a sequence of matrix **sizes** compatible for multiplication (though not the matrices themselves). The matrices will be multiplied in the usual way. Our task is to insert parentheses to make the multiplication fastest.

That is, we are given a sequence $a_0, a_1, a_2, \ldots$, that we interpret as describing a multiplication $A_0 A_1 A_2 \cdots$, where A_0 has a_0 rows and a_1 columns, A_1 has a_1 rows and a_2 columns, etc. If we choose parentheses $(A_0 A_1) A_2$, the cost is $a_0 a_1 a_2$ to multiply $A_0 A_1$ and get a a_0-by-a_2 matrix A', plus $a_0 a_2 a_3$ to multiply $A' A_2$. On the other hand, parenthesization $A_0(A_1 A_2)$ has cost $a_0 a_1 a_3 + a_1 a_2 a_3$.

As a more concrete example, on input $(3, 1, 4, 1)$,

$$\left(\begin{bmatrix}\cdot\\ \cdot\\ \cdot\end{bmatrix}\begin{bmatrix}\cdot & \cdot & \cdot & \cdot\end{bmatrix}\right)\begin{bmatrix}\cdot\\ \cdot\\ \cdot\\ \cdot\end{bmatrix}$$

has cost $3 \cdot 1 \cdot 4 + 3 \cdot 4 \cdot 1 = 24$ but

$$\begin{bmatrix}\cdot\\ \cdot\\ \cdot\end{bmatrix}\left(\begin{bmatrix}\cdot & \cdot & \cdot & \cdot\end{bmatrix}\begin{bmatrix}\cdot\\ \cdot\\ \cdot\\ \cdot\end{bmatrix}\right)$$

has cost $3 \cdot 1 \cdot 1 + 1 \cdot 4 \cdot 1 = 7 < 24$, so the latter parenthesization is preferable.

Give an algorithm that takes time polynomial in the number n of matrices and finds the best parenthesization.

Solution

We solve the problem and store results for each contiguous subsequence of matrices from the i'th to the j'th, in order of increasing $j - i$. If $j = i + 1$, the only choice gives cost $a_i a_{i+1} a_{i+2}$. For larger $j - i$, try all $j - i$ highest-level parenthesization into two matrices $(A_i \cdots A_k)(A_{k+1} \cdots A_j)$, and use the precomputed results for the two matrices, in constant time. There are at most n choices for k to consider for each of at most n^2 subsequences $i \ldots j$. This gives runtime $O(n^3)$.

Mathematical concepts: Dynamic programming