AIM Qualifying Review Exam: Probability and Discrete Mathematics

August 19, 2022

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

In a game, our goal is to determine exactly an unknown integer, $1, 2, \ldots, 1024 = 2^{10}$, chosen uniformly at random, by asking yes/no questions.

(a) What is the expected number of questions needed if we ask: Is it 1? Is it 2? (asking integers sequentially).

(b) What is the expected number of questions if we eliminate exactly half the possibilities with each question?

Show work. (Hint: Consider cases smaller than 1024 to check for off-by-1 errors.)

Problem 1

Problem 1

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Problem 2

Alice and Bob agree to meet at M36 coffee shop, but each arrives independently at random between noon and 1pm. What is the probability that the first to arrive has to wait more than 10 minutes?

Problem 2

Problem 2

Problem 2

Problem 3

Find the number of integers between 1 and 24,000, inclusive, that are not divisible by 4, 5, or 6.

Problem 3

Problem 3

Problem 3

Problem 4

Show that, for any sequence of n^2+1 distinct real numbers $a_1, a_2, \ldots, a_{n^2+1}$, either there is an increasing subsequence of length $n+1$ or a decreasing subsequence of length $n+1$.

Note: A subsequence is gotten by deleting zero or more elements from the original sequence, while retaining the order of the remaining terms. "Increasing" or "decreasing" refers to values. For example, given sequence $(5, 1, 2, 9, 8)$, the sequence $(5, 2)$ is a subsequence, gotten by striking 1, 9, 8 and leaving 5 and 2 in their original order with 5 preceding 2. It is decreasing, since $2 < 5$.

Problem 4

Problem 4

Problem 4

Problem 5

We are given a sequence of matrix **sizes** compatible for multiplication (though not the matrices themselves). The matrices will be multiplied in the usual way. Our task is to insert parentheses to make the multiplication fastest.

That is, we are given a sequence $a_0, a_1, a_2, \ldots$, that we interpret as describing a multiplication $A_0 A_1 A_2 \cdots$, where A_0 has a_0 rows and a_1 columns, A_1 has a_1 rows and a_2 columns, etc. If we choose parentheses $(A_0 A_1)A_2$, the cost is $a_0 a_1 a_2$ to multiply $A_0 A_1$ and get a a_0-by-a_2 matrix A', plus $a_0 a_2 a_3$ to multiply $A' A_2$. On the other hand, parenthesization $A_0(A_1 A_2)$ has cost $a_0 a_1 a_3 + a_1 a_2 a_3$.

As a more concrete example, on input $(3, 1, 4, 1)$,

$$\left(\begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \end{bmatrix} \right) \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

has cost $3 \cdot 1 \cdot 4 + 3 \cdot 4 \cdot 1 = 24$ but

$$\begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix} \left(\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \right)$$

has cost $3 \cdot 1 \cdot 1 + 1 \cdot 4 \cdot 1 = 7 < 24$, so the latter parenthesization is preferable.

Give an algorithm that takes time polynomial in the number n of matrices and finds the best parenthesization.

Problem 5

Problem 5

Problem 5