AIM Qualifying Review Exam in Differential Equations & Linear Algebra

August 2021

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

Let $\mathbf{A} = \begin{bmatrix} 1 & c \\ d & 1 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Consider the problem of minimizing $\|\mathbf{Ax} - \mathbf{b}\|_2$ over $\mathbf{x} \in \mathbb{R}^2$.

(a) For which values of c and d is there a unique solution for $\mathbf{x}$?

(b) When there is not a unique solution, express the set of $\mathbf{x}$ that achieve the minimum possible value of $\|\mathbf{Ax} - \mathbf{b}\|_2$ in terms of only one of c or d, but not both.

Solution

(a) If A has full column rank, 2 in this case, the unique solution is given by $\mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$. (The inverse in this expression exists since $\mathbf{A}$ has only nonzero singular values. The uniqueness is shown in standard linear algebra texts). Here $\mathbf{A}$ has full column rank iff the two columns are independent, i.e. $cd \neq 1$.

(b) We write $d = 1/c$, and minimize

$$\left\| \begin{bmatrix} 1 & c \\ 1/c & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 \\ 1/c \\ 0 \end{bmatrix} \begin{bmatrix} x + cy \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\|.$$

The unique solution formula in part a above can be applied only to the minimization of the second expression, since the leading matrix in the first expression does not have full column rank. Applying the solution formula to minimize the second expression, we have

$$x + cy = \frac{1 + 1/c}{1 + 1/c^2}.$$

The set of solutions is therefore

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1+1/c}{1+1/c^2} \\ 0 \end{bmatrix} + y \begin{bmatrix} -c \\ 1 \end{bmatrix}.$$

where y is a free parameter. A similar approach gives the solution in terms of d.

Problem 2

(a) Let V and W be two linear subspaces of $\mathbb{R}^n$. Prove or disprove: Both the union and intersection of V and W are linear subspaces of $\mathbb{R}^n$.

(b) What is the dimension of the set of vectors $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ that are simultaneously solutions of

$$\begin{bmatrix} 1 & 2 & 2 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 2 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad ?$$

Solution

(a) The assertion is false, because the intersection is a subspace but not the union. To prove the intersection is a subspace we show that $V \cap W$ is closed under addition and under scalar multiplication (i.e. under linear combination), and contains the zero vector. Clearly 0 is in both V and W, so it's in the intersection. Since V and W are separately closed under linear combination, a linear combination of elements in $V \cap W$ is in both V and W separately. So it is in $V \cap W$. To show the union is not a subspace we take V to be the span of $\mathbf{e}_1$, the first unit vector in the standard basis, and W to be the span of $\mathbf{e}_2$. $V \cup W$ is not closed under addition, as $\mathbf{e}_1 + \mathbf{e}_2 \notin V \cup W$.

(b) The set of $\{[x, y, z]^T\}$ that are solutions to the first equation is a one dimensional subspace, that which is orthogonal to the two-dimensional space spanned by $[1, 2, 2]^T$ and $[3, 0, 1]^T$. Thus $\{[x, y, z]^T\} = k\{[a, b, c]^T\}$ for some vector $\{[a, b, c]^T\}$ and $k \in \mathbb{R}$. Likewise, $\{[x, y, w]^T\} = k\{[a, b, c]^T\}$, $k \in \mathbb{R}$. So $\{[x, y, z, w]^T\} = k\{[a, b, c, c]^T\}$, $k \in \mathbb{R}$, a one-dimensional subspace.

Problem 3

Find a constant C such that the differential equation

$$\frac{d^2y}{dx^2} + y = C \cos 2x + x^3 + \sin x$$

has a solution that obeys

$$y(0) = y(\pi) = 0.$$

Solution

First we find the general solution as a sum of the complementary solution y_c and a particular solution y_p. y_c solves

$$\frac{d^2y}{dx^2} + y = 0,$$

so $y_c = A\cos x + B\sin x$. We guess y_p according to the form of the right hand side, as a sum of a polynomial of degree 3 and a linear combination of $\sin x$, $\cos x$, $\sin 2x$, $\cos 2x$, multiplied by powers of x as needed to avoid overlap with the terms in y_c. Therefore $y_p = Dx^3 + Ex^2 + Fx + G + H\cos 2x + I\sin 2x + x(J\cos x + K\sin x)$. Plugging in y_p and equating coefficients, we find $D = 1$, $E = 0$, $F = -6$, $G = 0$, $H = -C/3$, $I = 0$, $J =$ -1/2, and $K = 0$. So $y = A\cos x + B\sin x + x^3 - 6x - \frac{C}{3}\cos 2x - \frac{x}{2}\cos x$. Setting $y(0) = y(\pi) = 0$ gives two equations for the unknowns A and C. We find $A = \frac{\pi^3 - 6\pi + \pi/2}{2}$ and $C = 3\frac{\pi^3 - 6\pi + \pi/2}{2}$. B is left undetermined.

Problem 4

(a) Find the general solution to the linear system

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{x}.$$

(b) Identify the terms in your solution that have the largest magnitudes in the limits $t \to +\infty$ and $t \to -\infty$.

Solution

(a) The matrix in the system (call it M) has eigenvalue 2 with multiplicity 1 and eigenvalue 1 with muliplicity 2. The eigenspace for 2 is one-dimensional, spanned by $v_A = [3, 8, 1]^T$, so one solution is $v_A e^{2t}$. The one-dimensional eigenspace of 1 is the span of $v_B = [0, 1, 0]^T$. As M is degenerate, one poses another solution in the form $(v_B t + v_C)e^t$. Plugging in, one has to solve $(M - I)v_C = v_B$. A solution is $v_C = [1/2, 0, 0]^T$, so the general solution can be written as the linear combination $Av_A e^{2t} + Bv_B e^t + C(tv_B + v_C)e^t$.

(b) In the limit $t \to +\infty$, the term $Av_A e^{2t}$ has the largest magnitude, assuming $A \neq 0$. In the limit $t \to -\infty$, the term $C(tv_B + v_C)e^t$, or more precisely $Ctv_B e^t$, has largest magnitude, assuming $C \neq 0$. It decays to 0 more slowly than the other terms.

Problem 5

Solve the PDE

$$\partial_{tt} u - \partial_{xx} u = 0$$

for $u(x, t)$ on the interval $0 \leq x \leq \pi$ for $t > 0$ with the initial and boundary conditions:

$$u(x, 0) = 1 \ , \ \partial_t u(x, 0) = 0 \ , \ 0 \leq x \leq \pi$$
$$u(0, t) = u(\pi, t) = 0 \ , \ t > 0.$$

Solution

Using separation of variables, $u(x, t) = X(x)T(t)$, we have

$$\frac{X''}{X} = \frac{T''}{T} = c.$$

for a constant c. With the given boundary conditions for X, nontrivial solutions are of the form $X = \sin kx$, $k = 1, 2, \ldots$, with $c = -k^2$. The corresponding solutions for T are $A_k \cos kt + B_k \sin kt$, and we write

$$u(x, t) = \sum_{k=1}^{\infty} (A_k \cos kt + B_k \sin kt) \sin kx \ , \ 0 \leq x \leq \pi \ , \ t > 0. \tag{1}$$

Applying the initial condition $\partial_t u(x,0) = 0$ we have that $B_k = 0$ for all k. Applying the initial condition $u(x,0) = 1$ we have

$$1 = \sum_{k=1}^{\infty} A_k \sin kx. \tag{2}$$

Multiplying both sides by $\sin mx$ and integrating from $x = 0$ to π, we find $A_m = 4/(m\pi)$ for m odd and 0 for m even. Therefore the solution can be written

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{\pi} \frac{\cos\left((2n+1)t\right) \sin\left((2n+1)x\right)}{2n+1}. \tag{3}$$