Advanced Calculus and Complex Variables (solution hints)
August 2021
For full credit, support your answers with appropriate explanations.
There are five problems, each worth 20 points.

1. (20 points) Let Q be the set of rational numbers. Give an example of a function $f : [0,1] \to \mathbb{R}$ that satisfies the following two criteria:

   (a) f must be continuous at $x \in [0,1] - Q$.

   (b) f must be discontinuous at $x \in [0,1] \cap Q$.

   Explain why f has the above two properties. Informal explanations will get full credit.

   Solution Let $q_1, q_2, \ldots$ be an enumeration of rationals in $[0,1]$. Define

   $$f(x) = \sum_{\{n | q_n \leq x\}} \frac{1}{2^n}$$

   for $x \in [0,1]$. Then f has the above two properties.

2. A function $f : [0,1] \to \mathbb{R}$ is said to be lower semicontinuous if for every sequence $x_1, x_2, \ldots$ in $[0,1]$ with

   $$x_* = \lim_{n \to \infty} x_n$$

   we also have

   $$f(x_*) \leq \liminf_{n \to \infty} f(x_n).$$

   The sequence of values

   $$g_n = \inf\{f(x_n), f(x_{n+1}), \ldots\}$$

   is obviously increasing and therefore has a limit as $n \to \infty$ (the limit can be finite or $+\infty$). The limit of $g_1, g_2, \ldots$ is by definition $\liminf_{n\to\infty} f(x_n)$.

   (a) (10 points) If $f : [0,1] \to \mathbb{R}$ is a lower semicontinuous function, prove that it attains its infimum. That means there exists $x_* \in [0,1]$ such that

   $$f(x) \geq f(x_*)$$

   for all $x \in [0,1]$.

   (b) (10 points) Give an example of an $f : [0,1] \to \mathbb{R}$ that is lower semicontinuous but does not attain its supremum.

   Solution: (a) Suppose $m = \inf\{f(x) | x \in [0,1]\}$. There must exist a sequence $x_1, x_2, \ldots$ such that $\lim_{n\to\infty} f(x_n) = m$. By the Bolzano-W property and by taking a subsequence if necessary, we may assume that $\lim_{n\to\infty} x_n = x_*$. It then follows that $f(x_*) \leq m$ from the definition of lower semicontinuity and because m is the inf we must have $m = f(x_*)$. This is one of many possible proofs. (b) Define $f(x) = x$ for $0 < x < 1$ and $f(0) = f(1) = -1$. The supremum, which is 1, is not attained by this lower-semi function.

3. Sketch closed and oriented curves γ in $\mathbb{C}$ such that the value of

$$\frac{1}{2\pi i}\int_\gamma \left(\frac{1}{z-1}+\frac{1}{z-2}\right) dz$$

is 0, 1, and -2, respectively.

4. The function $f(z) = \sqrt{1-z^2}$ has branch points at $z = \pm 1$ but nowhere else. In particular, $z = \infty$ is not a branch point. Thus, we may choose the branch cut to be the interval $(-1, 1)$ in the real line and specify the branch by requiring $f(i) = +\sqrt{2}$.

   (a) (5 points) For $f(z)$ as specified above, is $f(z)$ positive or negative "slightly above" the branch cut $(-1, 1)$. Here "slightly above" refers to the limiting value of $f(z)$ as a point on the branch cut is approached from above.

   (b) (15 points) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dz}{\sqrt{1-z^2}},$$

   where it is assumed that the path from $-\infty$ to ∞ is along the real line and slightly above the branch cut. The branch of $f(z) = \sqrt{1-z^2}$ is as specified above.

   Solution (a) $f(z)$ is positive slightly above the branch cut. This may be proved using a continuity argument by first letting z vary from i to 0 and then above the branch cut. (b) First argue that $f(z) \sim -iz$ for $|z|$ large as follows. For large iy, $y > 0$, a continuity argument from $z = i$ upwards shows that $f(iy) \sim y = -iz$. Again by continuity, we must have $f(z) \sim -iz$ for all z with large $|z|$. The value of the integral can then be shown to be equal to

$$\int_\gamma \frac{1}{-iz}\, dz$$

   where γ is the path Re^{it} with $t \in [0, \pi]$ and ~~counter~~clockwise, and in the limit $R \to \infty$. Thus the integral evaluates to π.

5. Consider the function

$$f(z) = \left(z - \frac{\pi}{2}\right)\sin \pi z.$$

   (a) (10 points) Evaluate the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz,$$

   with γ being the close curve $|z| = 2\pi$ oriented counter-clockwise.

(b) (10 points) Let γ_n be the close curve $|z| = n + \frac{1}{2}$ oriented counter-clockwise and define

$$I_n = \frac{1}{2\pi i} \int_{\gamma_n} \frac{z^2 f'(z)}{f(z)}\, dz.$$

Evaluate the limit

$$\lim_{n\to\infty} \frac{I_n}{n^3}.$$

Above $n \in \mathbb{Z}^+$ is assumed.

Solution (a) $z = 0, \frac{\pi}{2}, \pm 1, \ldots, \pm 6$ are the roots of $f(z) = 0$ inside γ. Thus the answer is 14. (b) First argue that

$$I_n = \frac{\pi^2}{4} + 2(1^2 + \cdots + n^2)$$

using residues and the argument principle. The limit must then be equal to $\frac{2}{3}$.