AIM Qualifying Exam: Advanced Calculus and Complex Variables

August 2021

There are five (5) problems in this examination, each worth 20 points.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

1. (20 points) Let Q be the set of rational numbers. Give an example of a function $f : [0, 1] \to \mathbb{R}$ that satisfies the following two criteria:

   a) f must be continuous at $x \in [0, 1] - Q$.

   b) f must be discontinuous at $x \in [0, 1] \cap Q$.

   Explain why f has the above two properties. Informal explanations will get full credit.

2. A function $f : [0,1] \to \mathbb{R}$ is said to be lower semicontinuous if for every sequence $x_1, x_2, \ldots$ in $[0,1]$ with

$$x_* = \lim_{n\to\infty} x_n$$

we also have

$$f(x_*) \leq \liminf_{n\to\infty} f(x_n).$$

The sequence of values

$$g_n = \inf\{f(x_n), f(x_{n+1}), \ldots\}$$

is obviously increasing and therefore has a limit as $n \to \infty$ (the limit can be finite or $+\infty$). The limit of $g_1, g_2, \ldots$ is by definition $\liminf_{n\to\infty} f(x_n)$.

a) (10 points) If $f : [0,1] \to \mathbb{R}$ is a lower semicontinuous function, prove that it attains its infimum. That means there exists $x_* \in [0,1]$ such that

$$f(x) \geq f(x_*)$$

for all $x \in [0,1]$.

b) (10 points) Give an example of an $f : [0,1] \to \mathbb{R}$ that is lower semicontinuous but does not attain its supremum.

3. Sketch closed and oriented curves γ in $\mathbb{C}$ such that the value of

$$\frac{1}{2\pi i}\int_{\gamma}\left(\frac{1}{z-1}+\frac{1}{z-2}\right)dz$$

is 0, 1, and -2, respectively.

4. The function $f(z) = \sqrt{1-z^2}$ has branch points at $z = \pm 1$ but nowhere else. In particular, $z = \infty$ is not a branch point. Thus, we may choose the branch cut to be the interval $(-1, 1)$ in the real line and specify the branch by requiring $f(i) = +\sqrt{2}$.

   a) (5 points) For $f(z)$ as specified above, is $f(z)$ positive or negative "slightly above" the branch cut $(-1, 1)$. Here "slightly above" refers to the limiting value of $f(z)$ as a point on the branch cut is approached from above.

   b) (15 points) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dz}{\sqrt{1-z^2}},$$

   where it is assumed that the path from $-\infty$ to ∞ is along the real line and slightly above the branch cut. The branch of $f(z) = \sqrt{1-z^2}$ is as specified above.

5. Consider the function

$$f(z) = \left(z - \frac{\pi}{2}\right) \sin \pi z.$$

a) (10 points) Evaluate the integral

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz,$$

with γ being the closed curve $|z| = 2\pi$ oriented counter-clockwise.

b) (10 points) Let γ_n be the closed curve $|z| = n + \frac{1}{2}$ oriented counter-clockwise and define

$$I_n = \frac{1}{2\pi i} \int_{\gamma_n} \frac{z^2 f'(z)}{f(z)}\, dz.$$

Evaluate the limit

$$\lim_{n\to\infty} \frac{I_n}{n^3}.$$

Above $n \in \mathbb{Z}^+$ is assumed.