

AIM Preliminary Exam: Probability and Discrete Mathematics

January 6, 2015

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

A traditional soccer ball is made from 12 black regular pentagons and 20 white regular hexagons. A dodecahedron is made from 12 regular pentagons and zero hexagons. Show that any ball made from regular pentagons and hexagons must have exactly 12 pentagons.



Problem 1

Problem 1

Problem 1

Problem 2

Consider a $1 \times n$ chessboard. Let h_n denote the number of colorings by maize and blue in which no two adjacent squares are maize. Formulate a natural recurrence relation satisfied by h_n , simplify the generating function $f(x) = \sum_{n=0}^{\infty} h_n x^n$ into a rational function of x , and solve for h_n . (You may use symbols for relevant constant quantities, such as ϕ_{\pm} for $\frac{-1 \pm \sqrt{5}}{2}$.)

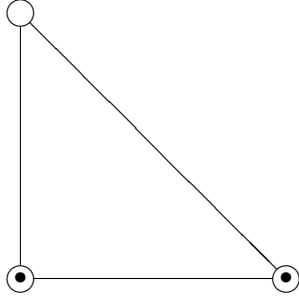
Problem 2

Problem 2

Problem 2

Problem 3

A **vertex cover** of an undirected graph $G = (V, E)$ is a set C of vertices such that each edge of G is incident on some vertex in C . Finding a vertex cover of minimal size is NP-hard. Below is a graph with an indicated vertex cover of size 2.



- (a) Show that the following algorithm achieves a 2-approximation. (That is, the algorithm finds a vertex cover C whose size is at most twice the size of the optimal vertex cover.)

$C \leftarrow \emptyset$
while some edge $\{u, v\}$ is uncovered
 Put u **and** v into C .

- (b) Show that the following algorithm can be very bad, by finding a graph G and choices for the **or** that makes the resulting C much bigger than optimal. How bad can the algorithm be?

$C \leftarrow \emptyset$
while some edge $\{u, v\}$ is uncovered
 Put u **or** v into C .

Problem 3

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Problem 4

Suppose n cards, numbered 1 to n , are shuffled. You are to make n guesses sequentially, where the i 'th one is a guess of the card in position i . Let N denote the number of correct guesses.

- (a) If you are not given any information about earlier guesses, show that any strategy gives the same value for $E[N]$. What is that value?
- (b) Suppose that, after each guess, you are shown the card that was in the position in question. Give an optimal strategy and find $E[N]$.
- (c) Suppose that you are told after each guess whether you are right or wrong. Consider the strategy that keeps guessing the same card until told it is correct and then changes to a new card. (This strategy, it turns out, optimizes $E[N]$.) Find $E[N]$.

Problem 4

Problem 4

Problem 4

Problem 5

The random variables X and Y have joint distribution function $f(x, y) = 12xy(1 - x)$ over $0 \leq x, y \leq 1$ (and zero otherwise).

- (a) Find $E[X]$.
- (b) Find $E[Y]$.
- (c) Find $\text{Var}(X)$.
- (d) Find $\text{Var}(Y)$.
- (e) Are X and Y independent?

Problem 5

Problem 5

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