AIM Preliminary Exam: Probability & Discrete Mathematics

January 8, 2012

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Suppose that each edge in the complete graph on n vertices is to be colored either red or blue. This problem concerns the following question: for a fixed integer $1 < k \leq n$, is there a coloring of the edges so that no set of k vertices has all of its $\binom{k}{2}$ edges the same color? Answer this question by stating (with proof) a relationship between n and k of the form $\binom{n}{k} < f(k)$, for some function f, that would guarantee there is at least one way of coloring the edges so that no set of k vertices has all of it edges the same color.

Problem 1

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Problem 2

Suppose that a random number generator produces an integer uniformly at random from the range $1, \ldots, 36,000,000$. What is the probability that the number chosen is divisible by one or more of 4, 6, and 9?

Problem 2

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Problem 3

Suppose that an airline ticket counter has n service agents and the time that agent i takes per customer has an exponential distribution with parameter θ_i (and these processing times are independent). You stand at the head of the line at time T_0 and all of the n agents are busy. The customers currently being served have been served for five minutes.

(a) What is the average time you wait for an agent? Prove your result.

(b) What is the probability that the jth agent is the first free agent? Prove your result.

Problem 3

Problem 3

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Problem 4

At the departmental tea, a starving student finds many more cookies than he had expected and he has to decide what to take (he doesn't have to pay for the cookies). His backpack will hold a total weight of at most $W = 5$ pounds. There are n types of cookies to pick from, of integer-valued weights $w_1, \ldots, w_n$ and dollar values $v_1, \ldots, v_n$. Devise a polynomial time (with respect to n with $W = 5$ fixed) algorithm and write it in pseudo-code to find the most valuable combination of cookies that he can fit into his backpack. There are two variants for which you should write pseudo-code:

(a) First, assume that the student can take multiple cookies of the same type; i.e., he can repeatedly choose the same type of cookie.

(b) Next, assume that the student *cannot* repeatedly choose the same type of cookie; refine your algorithm to include which items are chosen at each step.

Problem 4

Problem 4

Problem 4

Problem 5

Suppose that the arrival of customers to Starbucks is a Poisson process with rate λ. Suppose that exactly one customer arrives between opening time at 6:00am, and 7:00am. What is the probability that the single customer arrives before 6:45 am?

Problem 5

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