AIM Preliminary Exam: Probability and Discrete Mathematics

September 1, 2018

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

In the game of Nim, there are k piles of coins, initially with $n_1, n_2, \ldots$ coins, respectively. Two players, Alice and Bob, take turns removing any positive integer number of coins from any pile. Alice moves first. The winner is the play to take the last coin.

(a) Suppose there are $k = 2$ piles. For what values of n_1 and n_2 can Alice force a win, and when can Bob force a win? Give a winning strategy for each player for the corresponding initial configuration. (You may say, "the other case is similar," where true.)

Briefly give an invariant that holds as the game progresses, show that the winner can maintain that invariant by taking one or more coins from a single pile, and show that the game ends.

(b) Now suppose $k > 2$. Generalize the $k = 2$ case to give winning strategies for the appropriate player depending on the initial configuration. *Hint: Consider the vector S of mod 2 sums of the 2^0'th bits, the 2^1'st bits, etc., of $n_1, n_2, \ldots, n_k$.*

Problem 1

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Problem 2 Given a permutation $\langle i_1, i_2, \ldots, i_n \rangle$ on $\langle 1, 2, \ldots, n \rangle$, an inversion is a pair (j, k) such that $j < k$ but $i_j > i_k$.

How many permutations on $\langle 1, 2, \ldots, 6 \rangle$ are there with

(a) 13 inversions

(b) 14 inversions

(c) 15 inversions

(d) 16 inversions

Give examples.

Hint: Consider a_j, the number of k with $k > j$ but $i_k < i_j$, and consider the collection of sequences $\langle a_1, a_2, \ldots a_n \rangle$.

Problem 2

Problem 2

Problem 2

Problem 3

We are given a set of n Graduate Student Instructors and n courses that need a GSI. The GSIs rank the courses and the courses rank the GSIs (assume no ties). A *stable matching* is a bijection between GSIs and courses *without* the following situation: GSI g gets course c and g' gets c', but g prefers c' over c and c' prefers g over g'. (This is called an *instability* and might lead to an unauthorized trade.)

The Gale-Shapley algorithm is as follows, for set G of GSIs and set C of courses. During the course of the algorithm, GSIs and courses are *free* or *engaged* to exactly one course or GSI, respectively. GSIs may *propose* to teach some course.

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Initially all g ∈ G and c ∈ C are free
Initially no g ∈ G has proposed to any c ∈ C
While some free GSI g has not yet proposed to any course
    Let c be the favorite course of g to whom g has not yet proposed
    If c is free
        (g, c) become engaged
    Else let g' be such that (g', c) are engaged
        If c prefers g' to g
            (g', c) remains engaged and g remains free
        else (g, c) become engaged and g' becomes free
Return engaged pairs
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Note the following invariants hold:

1. For each course c, from the first proposal onward, c remains engaged and the sequence of GSI proposing to c gets better (in the ranking of c).
2. For each GSI g, the sequence of courses to whom g proposes gets worse (in the ranking of g) as the algorithm progresses.

Show:

(a) Give a (small) example showing that stable matchings need not be unique.

(b) Prove that the Gale-Shapley algorithm terminates after at most n^2 iterations of the while loop.

(c) Prove that the Gale-Shapley algorithm returns a perfect matching (i.e., a bijection between GSIs and courses).

(d) Prove that the perfect matching returned is stable.

Problem 3

Problem 3

Problem 3

Problem 4

A set of n dice is thrown. All those that land on six are put aside, and the others are again thrown. This is repeated until all the dice have landed on six. Let N denote the number of throws needed. (For instance, suppose that $n = 3$ and that on the inital throw exactly two of the dice land on six. Then the other die will be thrown and, if it lands on six, then $N = 2$.) Let $m_n = E[N]$.

(a) Compute m_1.

(b) Compute m_2.

Problem 4

Problem 4

Problem 4

Problem 5

Let X and Y be independent exponential random variables with respective rates λ and μ, i.e., X has density function

$$\begin{cases} \lambda e^{-\lambda x}, & x \geq 0; \\ 0 & x < 0 \end{cases}$$

and similarly for Y and μ.

(a) What is $E[X]$?

(b) Show that, conditional on $X > Y$, the random variables $M = \min(X, Y)$ and $D = X - Y$ are independent.

(c) What is $E[M]$?

Problem 5

Problem 5

Problem 5