AIM Preliminary Exam: Probability and Discrete Mathematics

September 1, 2014

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

The following definitions are standard.

A Reed-Solomon code has parameters q, k and n. Each codeword is associated with a polynomial p of dimension k (*i.e.*, degree at most $k - 1$), over the field of q elements. We assume q is prime and $k \leq n \leq q$. The codeword associated with p is the n-tuple $(p(0), p(1), p(2), \ldots, p(n-1))$.

An *error* in a codeword is the replacement of one component with another value of the right type (here, a number mod q). An *erasure* is the replacement of one component with "?" which is recognizably not of the right type. A code can *detect t errors* if there is an algorithm such that, whenever there is no error the algorithm says "correct" and whenever there is at least one and at most t errors the algorithm says "incorrect." A code can *correct t errors* if there's a decoding algorithm that, given any tuple of the form $x + e$ where x is a codeword and e is an error vector of at most t non-zeros, the algorithm returns x.

(a) Suppose $q = 11$, $k = 2$ and $n = 5$ (so each codeword is 5 points on a line mod 11). No justification is needed.

   (i) How many codewords are there?

  (ii) How many errors can be detected?

 (iii) How many erasures can be corrected?

 (iv) How many errors can be corrected?

(b) For general q, k, n, with $k \leq n \leq q$, answer and justify briefly:

   (i) How many codewords are there?

  (ii) How many errors can be detected?

 (iii) How many erasures can be corrected?

 (iv) How many errors can be corrected?

(c) Fix q and k. If we want to correct from up to a errors and up to b erasures (simultaneously), how big does n need to be in terms of q, k, a, b? Justify briefly.

You may assume that k points determine a polynomial of dimension k, modulo a prime q.

<u>**Problem 1**</u>

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Problem 1

Problem 2

H/T: Ross, 9e, p. 150

Two bags initially contain N marbles each. A marble is removed from one or the other bag, uniformly at random, until one of the bags is empty. At the time that one bag *first* becomes empty, what is the probability that the other bag contains exactly k marbles? You may express your answer using binomial coefficients and other standard symbols.

Problem 2

Problem 2

Problem 2

Problem 3

H/T: Ross, 9e, p. 236

Let X, Y, Z be independent and uniformly distributed over $(0, 1)$. Compute $\Pr(X \geq YZ)$.

Problem 3

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Problem 4

H/T CLRS

Suppose there is an agreed-upon exhange rate among all currencies, but banks charge (non-negative) fees for making exchanges. Each fee is expressed as a fraction of the amount exchanged.

Suppose we are given a table of fees for conversions among monetary currencies, of the form "Euro to Yen costs 5%." Some conversions are not possible; they have fee 100%. Let $f_{c,c'}$ denote the fee to convert currency c to currency c'.

Describe a polynomial-time algorithm that finds a table that lists, for each foreign currency c, the cost of the best way to convert US dollars into type c. Briefly show correctness and give a useful bound on the runtime (with brief proof). Note: it is not necessary to find the best algorithm; find a polynomial-time algorithm simple enough that you can give an overview of the proof. Also, it is not necessary to describe the exchange sequence to achieve the best cost, just the costs themselves. (It turns out that maintaining the best exchange sequences is a straightforward extension.)
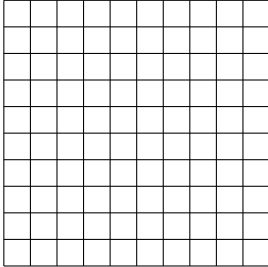
Problem 4

Problem 4

Problem 4

Problem 5

H/T: Cornelia Van Cott

Show that it is not possible to tile a 10×10 grid with 1×4 rectangles, some rotated by 90°. *Hint: Choose one of four colors for each of the diagonals of the 10×10 grid.*

Problem 5

Problem 5

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