

AIM Preliminary Exam: Probability and Discrete Mathematics

August 31, 2019

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Suppose that an ordinary deck of 52 cards is shuffled and the cards are then turned over one at time until the first ace appears. Given that the first ace is the 20'th card to appear, what is the probability that the card following it is

- (a) the ace of spades?
- (b) the two of clubs?

You need not simplify the expression, but justify your answer.

(Note: An ordinary deck has four aces and four twos, including one each of the ace of spades and two of clubs. Aces and twos are different cards.)

Problem 1

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Problem 2

If X and Y are independent random variables uniformly distributed on the interval $[0, 1]$, compute the joint density $f(u, v)$ and distribution $F(u, v)$ of $U = X + Y$ and $V = X/Y$ for the cases $0 \leq u \leq 1 \leq v$.

For full credit, sketch the unit square on the X - Y plane and indicate regions where various relevant events transpire. Also check F for the cases $u = 0, v = 1, v \rightarrow \infty$.

Problem 2

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Problem 3

Find integers a, b, c such that, for all m ,

$$m^3 = a \binom{m}{3} + b \binom{m}{2} + c \binom{m}{1}.$$

Then use the result to sum the series $1^3 + 2^3 + 3^3 + \cdots + m^3$ and express the sum as a linear combination of binomial coefficients of the form $\binom{m+1}{r}$.

Note that $\binom{m}{r} = \frac{m(m-1)\cdots(m-r+1)}{r!}$ is zero if m and r are positive integers and $r > m$.

Problem 3

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Problem 4

Determine the number of permutations of $\{1, 2, \dots, 8\}$ in which exactly four integers are in their natural positions.

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Problem 5

An array $A[1..n]$ of distinct numbers is *unimodal*, meaning the entries increase to a maximum $A[0..p]$ at index p , then decrease.

Give an algorithm to find the maximum, p , of a unimodal array that makes $O(\log(n))$ queries to the array. Analyze the runtime formally, by giving and solving a recurrence on the runtime $T(n)$ of the algorithm on arrays of length n .

Problem 5

Problem 5

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