AIM Qualifying Review Exam in Differential Equations & Linear Algebra

September 3, 2018

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

(a) (5 points) Consider the system of differential equations

$$\frac{du_1}{dt} = au_1 + bu_1^2 + cu_1u_2, \qquad \frac{du_2}{dt} = du_2 + eu_2^2 + fu_1u_2 \tag{1}$$

for populations $u_1 \geq 0$, $u_2 \geq 0$. What relations must the constants a through f obey for the system to describe a Predator-Prey model? I.e. one of the populations, a "prey" species, is consumed by the other, a "predator" species.

(b) (15 points) The system of differential equations

$$\frac{du_1}{dt} = u_1(2 - u_1 - u_2), \qquad \frac{du_2}{dt} = u_2(3 - 2u_1 - u_2) \tag{2}$$

describes populations $u_1 \geq 0$, $u_2 \geq 0$. Explain why these equations make it mathematically possible, but extremely unlikely, for both species to survive.

Problem 1

Problem 1

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Problem 2

(a) (10 points) Determine the stability properties of the origin for

$$\frac{dx}{dt} = -x^3 + 2y^3, \quad \frac{dy}{dt} = -2xy^2 \tag{3}$$

Hint: Use $V(x, y) = ax^2 + cy^2$.

(b) (10 points) Show that the origin is unstable for

$$\frac{dx}{dt} = x^3 - y^3, \quad \frac{dy}{dt} = 2xy^2 + 4x^2y + 2y^3. \tag{4}$$

Problem 2

Problem 2

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Problem 3

(a) What is the rank of the matrix $A = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 5 \end{pmatrix}$?

(b) Find a basis for the null space of A^T (A^T is the transpose of A).

(c) Determine all values of r for which the system

$$\begin{aligned} x - y &= 1 \\ 2x + y &= r \\ 3x + 5y &= -1 \end{aligned}$$

has a solution (x, y) and find one.

(d) Is the (x, y) solution you have found in part (c) unique? Why or why not?

(e) Does the null space of A have the same dimension as the null space of A^T? Explain your answer.

Problem 3

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Problem 4

Consider the matrix $B = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$.

(a) (8 points) Find a 2-by-2 matrix W such that $W^{-1}BW = D$ where D is a diagonal matrix.

(b) (12 points) Find projection matrices P_1 and P_2 such that $B = \lambda_1 P_1 + \lambda_2 P_2$ where λ_1 and λ_2 are the eigenvalues of B, $P_1 P_2 = 0$ and $P_1 + P_2 = I$, the 2-by-2 identity. Note: a projection matrix P satisfies $P^2 = P$.

Problem 4

Problem 4

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Problem 5

Find the solution $u(x,y)$ to the following boundary-value problem:

$$\begin{aligned} \partial_{xx}u + \partial_{yy}u = 0 \ , \ 0 < x < 1, \ 0 < y < 1 \\ u(0,y) = 0 \ , \ u(1,y) = h(y) \ , \ 0 \le y \le 1 \\ u(x,0) = 0 \ , \ \partial_y u(x,1) = 0 \ , \ 0 \le x \le 1. \end{aligned}$$

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