AIM Qualifying Review Exam in Differential Equations & Linear Algebra

January 8, 2019

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

(a) (6 points) Consider the system of differential equations

$$\frac{dx}{dt} = -x + y + x^2, \quad \frac{dy}{dt} = y - 2xy.$$

Find a function $H(x, y)$ that remains constant in time along solution trajectories.

(b) (7 points) Solve the differential equation

$$\frac{dy}{dx} = \frac{y + x}{y - x}.$$

(c) (7 points) Find all initial conditions (x_0, y_0) (with $y(x_0) = y_0$) such that the solution $y(x)$ in part (b) exists and is unique in the interval $x_0 - \epsilon < x < x_0 + \epsilon$ for some $\epsilon > 0$.

Problem 1

Problem 1

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Problem 2

(a) (15 points) Find the solution of $y'' + 4y = 2\sin 2t, y(0) = 2, y'(0) = -1$.

(b) (5 points) Find the general solution of $t^2y'' + 4ty' + 2y = 0$ in an interval not containing $t = 0$.

Problem 2

Problem 2

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Problem 3

(a) (10 points) For which real symmetric matrices A is $A - A^3$ invertible? Why?

(b) (10 points) Find the determinant and characteristic polynomial of $A = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$.

Problem 3

Problem 3

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Problem 4

Let A be an n-by-n real square symmetric matrix. Prove or disprove the following statements:

(a) Suppose A is invertible. Then there is a number $\epsilon > 0$ such that $A + B$ is invertible for all n-by-n matrices B with entries less than ϵ in magnitude.

(b) Suppose A is *not* invertible. Then there is a number $\epsilon > 0$ such that $A + B$ is *not* invertible for all n-by-n matrices B with entries less than ϵ in magnitude.

Problem 4

Problem 4

Problem 4

Problem 5

(a) (12 points) Find the solution $u(x,t)$ to the following initial-boundary-value problem:

$$\partial_t u - \partial_{xx} u = \cos 2\pi x \ , \ 0 < x < 1, \ t > 0$$
$$\partial_x u(0,t) = 0 \ , \ \partial_x u(1,t) = 0 \ , \ t > 0$$
$$u(x,0) = \sin^2 3\pi x \ \ , \ 0 \leq x \leq 1.$$

(b) (8 points) Find the solution again when the PDE is changed to

$$\partial_t u - \partial_{xx} u = \cos 2\pi x \sin t$$

with the same initial and boundary conditions.

Problem 5

Problem 5

Problem 5