AIM Qualifying Review Exam in Differential Equations & Linear Algebra

September 5, 2015

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Let $\mathbf{B}_n$ be the n-by-n matrix, $n > 1$, with ones on the main diagonal, ones on the reverse main diagonal, and zeros elsewhere. For example,

$$\mathbf{B}_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \mathbf{B}_4 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \mathbf{B}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \ldots.$$

Find a complete set of eigenvalues and eigenvectors of $\mathbf{B}_n$ for:

(a) an arbitrary odd integer n and

(b) an arbitrary even integer n.

Hint: Start with $n = 3$ and 4.

Problem 1

Problem 1

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Problem 2

Let $\mathbf{A} = \mathbf{a}\mathbf{b}^T + \mathbf{c}\mathbf{d}^T + \mathbf{e}\mathbf{f}^T$, where $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$, and $\mathbf{f}$ are real column vectors with five components, and none are the zero vector. Here ^{T} denotes transpose.

(a) Which of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$, and $\mathbf{f}$ must lie in the range of $T_1(\mathbf{x}) = \mathbf{A}\mathbf{x}$?

(b) What are the minimum and maximum possible values of the nullity of $\mathbf{A}$? The nullity is the dimension of the nullspace of $\mathbf{A}$.

(c) Find a condition on $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$, and $\mathbf{f}$ which implies that $\mathbf{A}$ has rank 1.

(d) Which of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$, and $\mathbf{f}$ must lie in the range of $T_2(\mathbf{x}) = \mathbf{A}^T\mathbf{x}$?

Problem 2

Problem 2

Problem 2

Problem 3

(a) Find the form of a particular solution to the differential equation $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = \cos x$. You may leave arbitrary constants in your solution.

(b) Compute the entries of the matrix $e^{\mathbf{J}t}$, where $\mathbf{J} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Problem 3

Problem 3

Problem 3

Problem 4

Consider the ODE system

$$\frac{dx}{dt} = x - 2xy \tag{1}$$

$$\frac{dy}{dt} = -y + xy. \tag{2}$$

(a) Find all critical points.

(b) For each critical point, find its type and stability. Sketch solution trajectories in the neighborhood of each critical point.

(c) Draw a phase portrait for the system in all four quadrants of the x-y plane.

(d) List the possible limiting behaviors as $t \to \infty$ of solutions in the first quadrant only.

Problem 4

Problem 4

Problem 4

Problem 5

Consider the heat equation with a source term, $\partial_t u = \kappa^2 \partial_{xx} u + \sin(5\pi x)$ on the domain $x \in (0,1)$ and $t > 0$. Let the boundary conditions be

$$u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0 \quad \text{for all } t > 0. \tag{3}$$

Let the initial condition be

$$u(x,0) = \sin(\pi x) \quad \text{for} \quad x \in (0,1). \tag{4}$$

Find the solution for all $\kappa > 0$.

In the limit $t \to \infty$, what is $u(x,t)$?

Problem 5

Problem 5

Problem 5