AIM Preliminary Exam: Differential Equations & Linear Algebra

September 1, 2012

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Consider the differential equation $x'' + 2x' + x = f(t)$ for an unknown function $x = x(t)$.

(a) In the special case that $f(t) := -2\sin t$, find the solution $x(t)$ satisfying the initial conditions $x(0) = 0, x'(0) = 1$.

(b) In the case of a general continuous forcing function $f(t)$, find a kernel function $k(t)$ such that the expression

$$x(t) = \int_0^t k(t-s)f(s)ds$$

is the solution of the equation satisfying the initial conditions $x(0) = 0, x'(0) = 0$.

Problem 1

Problem 1

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Problem 2

Consider the Laplace equation in polar coordinates, $\Delta u = \dfrac{1}{r}\partial_r(r\partial_r u) + \dfrac{1}{r^2}u_{\theta\theta} = 0$, in the domain $\{1 < r < 2, 0 < \theta < \pi\}$, with boundary conditions

$$\begin{aligned} u(r,0) = u(r,\pi) &= 0, && 1 < r < 2, \\ u(1,\theta) &= \sin\theta, && 0 < \theta < \pi, \\ u(2,\theta) &= 0, && 0 < \theta < \pi. \end{aligned}$$

(a) Find the solution $u(r,\theta)$.

(b) Sketch the domain and indicate the locus of points at which the solution attains its maximum and minimum values.

Problem 2

Problem 2

Problem 2

Problem 3

Consider the damped wave equation $u_{tt} + 4u_t = u_{xx}$ on the domain $0 < x < \pi$ with boundary conditions $u(0,t) = u(\pi,t) = 0$.

(a) Find the solution $u(x,t)$ subject to the initial conditions: $u(x,0) = \sin x, u_t(x,0) = 0$.

(b) Find the solution $u(x,t)$ subject to the initial conditions: $u(x,0) = \sin 3x, u_t(x,0) = 0$.

(c) In which of the above two cases does the solution $u(x,t)$ converge to zero faster?

Problem 3

Problem 3

Problem 3

Problem 4

(a) Let $A = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$. Find $\lim_{n\to\infty} A^n$.

(b) Consider the system of differential equations,

$$3x' = 2x + y, \quad 3y' = x + 2y.$$

Find the solution $x(t), y(t)$ satisfying the initial conditions $x(0) = 1, y(0) = 1$.

Problem 4

Problem 4

Problem 4

Problem 5

Consider the three linear equations on two unknowns x_1 and x_2 given by

$$x_1 + x_2 = 1, \quad x_1 - x_2 = 0, \quad x_1 + 2x_2 = 1.$$

(a) Find the least squares solution of the system.

(b) Each equation of the system defines a line in the (x_1, x_2)-plane. Sketch the lines and indicate the position of the least squares solution.

(c) Consider a general linear system of m equations in n variables, with $m > n$. If the system is written in the form $Ax = b$, where $A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n, b \in \mathbb{R}^m$, what condition(s) on the matrix A ensure that the least squares solution is unique?

Problem 5

Problem 5

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