AIM Qualifying Review Exam in Differential Equations & Linear Algebra

September 2, 2019

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

Consider the system of differential equations

$$\frac{dx}{dt} = \sin y, \quad \frac{dy}{dt} = \sin x.$$

(a) (3 points) Find all critical (i.e. equilibrium) points of the system.

(b) (6 points) Find the corresponding linear system near each critical point.

(c) (4 points) Find the eigenvalues for each linear system.

(d) (4 points) Draw a phase portrait of representative solution trajectories near all critical points in the region $\{-5 < x < 5, -5 < y < 5\}$. Use arrows to label the directions of the solutions along the trajectories as time increases.

(e) (3 points) Find a function $H(x, y)$ that remains constant in time along solution trajectories.

Problem 1

Problem 1

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Problem 2

(a) (14 points) Find the general solution to

$$\frac{d^5y}{dt^5} + \frac{d^2y}{dt^2} = \cos t.$$

(b) (2 points) Find an initial condition for which the solution to part (a) remains bounded as $t \to +\infty$.

(c) (2 points) Find an initial condition for which the solution to part (a) becomes unbounded as $t \to +\infty$.

(d) (2 points) Find a function with the following property: If the function is entered in place of $\cos t$ on the right hand side of part (a), the solution $y \to -\infty$ as $t \to +\infty$ *for all initial conditions*.

Problem 2

Problem 2

Problem 2

Problem 3 Consider $P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

(a) (7 points) Find P^2, P^{999}, and the determinant of P.

(b) (7 points) Find the eigenvectors and eigenvalues of P.

(c) (4 points) Find a singular value decomposition of P and a QR factorization of P.

(d) (2 points) For which $v \in \mathbb{C}^4$ does $\lim_{k\to\infty} P^k v$ exist?

Problem 3

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Problem 4

(a) (4 points) Find a matrix A (not necessarily square) that has a left inverse (i.e. B with $BA =$ identity) but not a right inverse.

(b) (4 points) Let $B = CD$, where C is a real 3-by-2 matrix and D is a real 2-by-3 matrix, so B is 3-by-3. Are there C and D such that B is invertible?

(c) (4 points) Let $E = FG$, where F is a real 2-by-3 matrix and G is a real 3-by-2 matrix. Are there F and G such that E is invertible?

(d) (4 points) Let $\mathbb{C}^4$ have a three-dimensional subspace $\mathbf{S}$. True or false: Every basis for $\mathbf{S}$ can be extended to a basis for $\mathbb{C}^4$ by adding one more vector.

(e) (4 points) Let $\mathbb{C}^4$ have a three-dimensional subspace $\mathbf{S}$. True or false: Every basis for $\mathbb{C}^4$ can be reduced to a basis for $\mathbf{S}$ by removing one vector.

Problem 4

Problem 4

Problem 4

Problem 5

(a) (15 points) Find the form of the general solution $u(x, y)$ to the following equation:

$$\partial_{xx}u - \partial_{yy}u = \sin 2\pi x \ , \ 0 < x < 1, \ 0 < y < 1$$

with the following boundary conditions:

$$\begin{aligned} u(x,0) &= u(x,1) \ , \ 0 \le x \le 1 \\ u(0,y) &= u(1,y) \ \ , \ 0 \le y \le 1. \end{aligned}$$

(b) (5 points) What is the smallest value of

$$\int_0^1 \int_0^1 (\partial_x u)^2 + (\partial_y u)^2 \, dx \, dy.$$

attained by the solutions to part (a)?

Problem 5

Problem 5

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