AIM Preliminary Exam: Advanced Calculus & Complex Variables

January 8, 2011

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

(a) Determine the values of the parameter $p > 0$ for which the following series converges:

$$S_a := \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^p + 1}.$$

(b) Repeat for the following series:

$$S_b := \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^p + (-1)^n}.$$

Problem 1

Problem 1

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Problem 2

(a) Suppose that f is analytic in a disk centered at the origin of radius R. Let $z \neq 0$ be fixed, and suppose that C is a positively-oriented circle about the origin of radius $r < \min(R, |z|)$. For integer $n \geq 0$, evaluate

$$F_n(z) := -\frac{z^{n+1}}{2\pi i} \oint_C \frac{s^{-n-1} f(s)\, ds}{s - z}$$

and identify the result.

(b) Suppose that $k \in \mathbb{R}$. Evaluate as a function of k the integral

$$I(k) := \int_{-\pi/2}^{\pi/2} e^{ik \tan(\phi)}\, d\phi.$$

Problem 2

Problem 2

Problem 2

Problem 3 Consider the power series

$$f(z) := \sum_{n=0}^{\infty} z^{2^n}.$$

(a) Prove that this series converges for $|z| < 1$.

(b) Prove that for $|z| < 1$,

$$f(z^{1/2^m}) = z^{1/2} + z^{1/4} + z^{1/8} + \cdots + z^{1/2^m} + f(z)$$

holds for any $m = 1, 2, 3, \ldots$.

(c) Prove that

$$\lim_{z \uparrow 1} f(z) = +\infty.$$

(Hint: consider a sequence of positive real numbers of the form $z = r^{1/2^m}$, $m = 0, 1, 2, \ldots$ with $0 < r < 1$.)

(d) Prove that for $|z| < 1$,

$$f(z) = z + z^2 + z^4 + \cdots + z^{2^{m-1}} + f(z^{2m})$$

holds for any $m = 1, 2, 3, \ldots$.

(e) Prove that

$$\lim_{r \uparrow 1} |f(re^{i\theta})| = +\infty$$

for a dense set of angles $\theta \in (-\pi, \pi)$. This proves that f is an analytic function for $|z| < 1$ with no analytic continuation beyond the unit circle along any path at all; that is, the unit circle is a *natural boundary* for f.

Problem 3

Problem 3

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Problem 4

Suppose that $\vec{u}$ and $\vec{v}$ are vector fields on $\mathbb{R}^3$ with smooth components. Suppose also that $\vec{v}$ vanishes identically outside of a sphere of large radius R, and that $\text{div}(\vec{u}) = 0$ and $\text{curl}(\vec{v}) = \vec{0}$ hold as identities on $\mathbb{R}^3$. Use the Divergence Theorem to calculate the integral

$$I := \iiint_{\mathbb{R}^3} \vec{u}(\vec{x}) \cdot \vec{v}(\vec{x}) \, dV(\vec{x}).$$

Give full details of your work.

Problem 4

Problem 4

Problem 4

Problem 5

Being careful about all details, prove the identity

$$\int_0^1 \frac{dx}{x^x} = \sum_{n=1}^{\infty} \frac{1}{n^n}.$$

Hint: consider expanding $1/x^x$ in an appropriately convergent series.

Problem 5

Problem 5

Problem 5