

AIM Qualifying Review Exam in Advanced Calculus & Complex Variables

September, 2016

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Consider the class of functions $f : \mathbb{R} \rightarrow \mathbb{R}$.

- (a) Give an example of a function which is not continuous at any point. Justify your answer.
- (b) Give an example of a function which is continuous at every irrational point but not at any rational point. Justify your answer.

Problem 1

Problem 1

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Problem 2

Consider the geometric series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \dots$$

(a) Find a formula for the remainder $R_{2n}(x)$ in

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \dots + (-1)^{n-1}x^{2n-2} + R_{2n}(x)$$

(b) Use that formula to prove that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Problem 2

Problem 2

Problem 2

Problem 3 Find the image of the following two regions under the map

$$w = \frac{1}{2} \left(z + \frac{1}{z} \right)$$

in the w -plane:

(a) The annulus $\left\{ z \mid \frac{1}{2} < |z| < 1 \right\}$.

(b) The quarter disc $\left\{ z \mid |z| < 1 \text{ and } x > 0, y > 0 \right\}$. Here $z = x + iy$.

Problem 3

Problem 3

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Problem 4 Evaluate the integral

$$\int_0^{\infty} \frac{\log x}{1+x^2} dx$$

using residues.

Problem 4

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Problem 5 Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2}$$

using the integral $\int \frac{1}{1+z^2} \cot \pi z$ over a suitable contour.

Problem 5

Problem 5

Problem 5