AIM Preliminary exam: Advanced Calculus & Complex Variables

September 1, 2018

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

(20 points) Consider a sequence $\{c_q\}_{q \geq 1}$ with terms taking values in the set $\{0, 1\}$ and define the sequence of finite binary fractions $\{x_j\}_{j \geq 1}$ with terms

$$x_j = 0.c_1 c_2 \dots c_j = \sum_{q=1}^{j} \frac{c_q}{2^q}.$$

Prove that the sequence $\{x_j\}_{j \geq 1}$ is convergent.

Problem 1

Problem 1

Problem 1

Problem 2

(20 points) Prove that the series $\sum_{j=1}^{\infty} \frac{1}{j(j+1)}$ converges and calculate its sum. Use this series to prove that $\sum_{j=1}^{\infty} \frac{1}{j^2+1}$ converges and that its sum lies in an interval (c_1, c_2) with constants c_1 and c_2 satisfying $0 < c_1 < c_2 < \infty$ that you should determine.

Problem 2

Problem 2

Problem 2

Problem 3

(20 points)

1. Consider a sequence $\{s_n\}_{n\geq 1}$ of positive numbers satisfying $\lim_{n\to\infty} \frac{s_{n+1}}{s_n} < 1$. Prove that $\lim\limits_{n\to\infty} s_n = 0$.

2. Consider the sequence of functions $\{f_n(x)\}_{n\geq 1}$ defined for $x \in [0,1]$ by

$$f_n(x) = n^2 x^n (1-x).$$

   Does this sequence have a pointwise limit as $n \to \infty$? If yes, is the convergence to this limit uniform?

Problem 3

Problem 3

Problem 3

Problem 4

(20 points) Let Ω denote a region (i.e., an open connected set) in the complex plane. In the following, u and v denote real valued functions with arguments (x, y), where $z = x + iy \in \Omega$.

1. Show that if $f = u + iv$ is analytic in Ω and u is constant, then f is constant.
   Hint: You can use the fact that any two points in Ω can be connected by a polygonal path with horizontal and vertical segments.

2. Show that if $f = u + iv$ is analytic in Ω and $|f|$ is constant, then f is constant.

3. Show that a non-constant analytic function cannot map Ω into a straight line or a circular arc.

Problem 4

Problem 4

Problem 4

Problem 5

(20 points)

1. Let x, y, z denote the coordinates in $\mathbb{R}^3$. Find the volume of the set $\Omega \subset \mathbb{R}^3$ that is bounded by the surface $x^2 + y^2 + z = 6$, and the following 5 planes:

$$\begin{aligned} P_1 &= \{(x, y, z) \in \mathbb{R}^3 : \ x = 1\}, \\ P_2 &= \{(x, y, z) \in \mathbb{R}^3 : \ y = 2\}, \\ P_3 &= \{(x, y, z) \in \mathbb{R}^3 : \ x = 0\}, \\ P_4 &= \{(x, y, z) \in \mathbb{R}^3 : \ y = 0\}, \\ P_5 &= \{(x, y, z) \in \mathbb{R}^3 : \ z = 0\}. \end{aligned}$$

2. Show that $f(x, y) = x^2 + 2y^2$ has a maximum on the circle $x^2 + y^2 = 1$. At what point is it achieved? If there are multiple such points, list them all.

Problem 5

Problem 5

Problem 5