AIM Preliminary Exam: Advanced Calculus & Complex Variables

September 3, 2011

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Consider the series

$$S(x) := \sum_{n=0}^{\infty} \frac{1}{x^2 - n^2}, \quad x \in \mathbb{R}.$$

(a) Determine all $x \in \mathbb{R}$ for which this series converges.

(b) Let $F \subset \mathbb{R}$ be the set of points x where the series *fails* to converge. For $M > 0$, consider the set $Y_M := \{x \in \mathbb{R},\ x \notin F,\ |x| < M\}$. Let $M > 0$ be fixed (arbitrary). Does the series converge uniformly on Y_M? Prove or disprove.

Problem 1

Problem 1

Problem 1

Problem 2

Suppose a monkey roams over all of a certain simply-connected tropical island I with smooth shoreline S. Thus the probability of finding the monkey in a region R of the island is

$$\mathbb{P}(\text{monkey in } R) = \frac{\text{Area}(R)}{\text{Area}(I)},$$

implying a uniform (constant) probability density $p(x,y) = 1/\text{Area}(I)$. In probability theory, the expected position of the monkey on the island is given by the coordinates $(\mathbb{E}[x], \mathbb{E}[y])$ defined relative to the probability density $p(x,y)$ by

$$\mathbb{E}[x] := \iint_I p(x,y)x\,dx\,dy \quad \text{and} \quad \mathbb{E}[y] := \iint_I p(x,y)y\,dx\,dy.$$

Express the monkey's expected position *only in terms of the shoreline* S.

Problem 2

Problem 2

Problem 2

Problem 3

(a) Let C be the contour parametrized by the mapping $t \mapsto z(t) := 16\cos(\frac{1}{2}t) + i\sin(t)$, for t increasing from $-\pi$ to π. Sketch C.

(b) Explain the convergence (existence) of the contour integral

$$I := \int_C \frac{\log(z)\,dz}{(z-1)(z-2)(z-4)(z-8)}$$

where $\log(z)$ is the principal branch of the natural logarithm, and where C is the contour described in part (a).

(c) Evaluate I.

(d) Evaluate the real integral

$$J := \int_{-\infty}^{0} \frac{dx}{(x-1)(x-2)(x-4)(x-8)}$$

by relating J to I.

Problem 3

Problem 3

Problem 3

Problem 4

Consider the mapping $x+iy = z \mapsto w := 2z^{-1/2} - 1$, where $z^{-1/2}$ denotes the principal branch of the power function on the complex z-plane.

(a) Find the image under $w(z)$ of the unbounded region given by the inequality $x > 1 - \frac{1}{4}y^2$.

(b) Repeat for the unbounded region $x < 1 - \frac{1}{4}y^2$.

(c) Suppose we use the mapping $z \mapsto w = r+is$ to solve a mixed Dirichlet/Neumann problem for Laplace's equation on the domain $x > 1 - \frac{1}{4}y^2$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for} \quad x > 1 - \tfrac{1}{4}y^2,$$

subject to the inhomogeneous Dirichlet boundary condition

$$u(x,y) = \frac{1}{x^2+y^2} \quad \text{for} \quad x = 1 - \tfrac{1}{4}y^2 \text{ and } x < 0,$$

and the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0 \quad \text{for} \quad x = 1 - \tfrac{1}{4}y^2 \text{ and } x \geq 0$$

where $\partial/\partial n$ denotes the derivative in the direction normal to the boundary. We also require that $u(x,y) \to 0$ as $(x,y) \to \infty$. What are the corresponding boundary conditions satisfied by $u(x(r,s), y(r,s))$ when viewed as a function of $w = r + is$ on the boundary of the image domain found in part (a)? Why?

Problem 4

Problem 4

Problem 4

Problem 5

Let $f(z)$ be an entire function.

(a) Suppose that there is a constant $C > 0$ and nonnegative integer N such that the inequality $|f(z)| \leq C(1 + |z|^N)$ holds for all $z \in \mathbb{C}$. Prove directly that $f(z)$ is then a polynomial of degree at most N. Hint: consider the Cauchy integral formula for $f^{(N+1)}(z)$ with a circular contour centered at z having (large) radius R.

(b) Suppose that there is a constant $C > 0$ and nonnegative integer N such that $|f(z)| \leq C|z|^N$ holds for all $z \in \mathbb{C}$. Prove that $f(z) = f_0 z^N$ for some constant $f_0 \in \mathbb{C}$.

Problem 5

Problem 5

Problem 5