Random Walks on the Fundamental Group of the Once Punctured Torus

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Abstract

The purpose of this paper is to investigate the long term behavior of a random walk on the fundamental group of the once punctured torus; in particular, the average amount of time a random walk spends near the puncture. By introducing a systematic way of symbolically representing geodesics in the universal covering space of the punctured torus, we aim to better understand the asymptotic behavior of random walks. Using these symbolic codings of geodesics we conjecture that a random walk associated to a simple closed curve on the once punctured torus on average spends no time in the cusp.

1 Random Walks on Groups

1.1 Definitions

We are interested in studying random walks on groups, more specifically on free groups. In this section we will introduce the main definitions we will be using. First, recall the definitions of a stochastic process and a Markov Chain.

Definition 1. A **stochastic process** is a collection of random variables $\{X_t, t \in T\}$ defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where T is an index set. These random variables take values over a set E, called the state space.

Definition 2. The stochastic process $\{X_t, t \in T\}$ is called a **Markov Chain** if $\forall n \in \mathbb{N}$ and for every collection $x_0, x_1, ..., x_n \in E$, if $\mathbb{P}(X_0 = x_0, X_1 = x_1, ..., X_{n-1} = x_{n-1}) > 0$ then

$$\mathbb{P}(X_n = x_n | X_0 = x_0, X_1 = x_1, ..., X_{n-1} = x_{n-1}) = \mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1}).$$

Remark 1. We will call $P_{xy} := \mathbb{P} = (X_n = y | X_{n-1} = x)$ the transition probability form x to y.

We will use these definitions to define a random walk on a free group. First, let's remember the definition of a free group.

Definition 3. Given a finite set S, we define its inverse set as $S^{-1} := \{s^{-1} : s \in S\}$. No relation exists between the elements of these sets other than the relationship between an element and its inverse. A **free group on** S consist of all the reduced words formed by elements in $S \cup S^{-1}$. The group operation is defined by the juxtaposition of words with the appropriate cancellation of letters if required.

We now want to equip a free group with a metric. We will do so the following way:

Definition 4. Let G be the free group on the set S. The **word length** $l_s(g) = n$ where n is the smallest integer such that

$$g = s_1 s_2 ... s_n \qquad s_i \in S \cup S^{-1}.$$

We can define the **word metric** d_s on G as

$$d_s(g_1, g_2) = l_s(g^{-1} g_2).$$

Definition 5. Given a group G its corresponding **Cayley Graph** $C(G)$ is the graph with G as the set of vertices and the edge (g_1, g_2) exists if and only if $d_s(g_1, g_2) = 1$.

Remark 2. The Cayley Graph of a given group is not unique, it depends on the choice of the generating set S. It is unique up to quasi-isometries.

Definition 6. A **random walk on G** determined by a probability measure μ on G is a Markov chain with state space G and transition probabilities that don't change over time and are given by

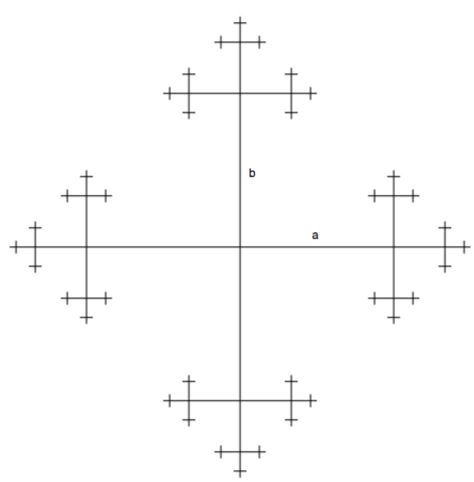
$$p(g \mid h) = \mu(h^{-1} g) \qquad g, h \in G.$$

Definition 7. A μ**-stationary measure** of a random walk on G is a probability measure ν on G such that

$$\mu * \nu(g) = \nu(g) \qquad \forall g \in G.$$

1.2 Free group on two generators $\mathbb{F}_2 = \langle a, b \rangle$

We will illustrate the previous concepts for the free group of two generators. The following figure is the ball of radius 4 of the Cayley Graph of $\mathbb{F}_2$.



We will consider the random walk on $\mathbb{F}_2$ given by the probability measure μ on $\mathbb{F}_2$ defined as follows

$$\mu(x) = \begin{cases} 1/4, & \text{if } x \in \{a, b, a^{-1}, b^{-1}\} \\ 0, & \text{otherwise.} \end{cases}$$

Notice that with this probability measure the random walk on the group is a random walk on the Cayley graph such that standing on each vertex there is equal probability to move to an adjacent vertex on the next step and probability 0 of moving anywhere else.

We will prove that

$$\nu(x) = \begin{cases} 1/(4 \cdot 3^{d_s(g,e)}), & \text{if } x \neq e \\ 1/4, & \text{if } x = e \end{cases}$$

is a μ-stationary measure.

Proof. If $x \neq e$ we have that

$$\begin{aligned} \mu * \nu(x) &= \sum_{g \in G} \mu(g)\nu(g^{-1}x) \\ &= \sum_{x \in \{a,b,a^{-1},b^{-1}\}} \frac{1}{4}\nu(g^{-1}x) \\ &= \frac{1}{4}\big(\nu(a \cdot x) + \nu(b \cdot x) + \nu(a^{-1} \cdot x)) + \nu(b^{-1} \cdot x)\big) \\ &= \frac{1}{4}\left(3 \cdot \frac{1}{4 \cdot 3^{d_s(x,e)}} + \frac{1}{4 \cdot 3^{d_s(x,e)-2}}\right) \\ &= \frac{1}{4 \cdot 3^{d_s(g,e)}}. \end{aligned}$$

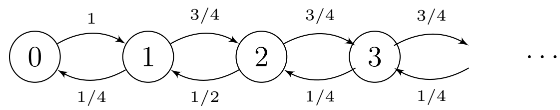
The forth equality is due to the fact that when composed with x the elements in $\{a, b, a^{-1}, b^{-1}\}$ increase the distance to e by one and one element decreases the distance by one, namely the element that is the inverse of the first letter of x.

On the other hand, if $x = e$, then

$$\begin{aligned} \mu * \nu(e) &= \sum_{g \in G} \mu(g)\nu(g^{-1}) \\ &= \sum_{k \in \{a,b,a^{-1},b^{-1}\}} \frac{1}{4}\nu(g^{-1}) \\ &= 4 \cdot \frac{1}{4} \cdot \frac{1}{4} \\ &= \frac{1}{4} \end{aligned}$$

We can conclide that $\mu * \nu(g) = \nu(g)$, $\forall g \in G$. That is, ν is a μ-stationary measure. $\square$

A random walk is called recurrent if the probability of returning to the origin an infinite number of times is equal to 1, otherwise it is called transient. Equivalently, a random walk is recurrent if and only if $\sum_{n=0}^{\infty} u_n$ diverges, where u_n is the probability of returning to the origin after n steps. We should observe that the Cayley graph of $\mathbb{F}_2$ can be seen as an infinite 4-valent tree. We will call the set of vertices that are at distance m from the origin level m. Because the random walk is symmetric given that you are at level $m > 0$ the probability of stepping back to level $m - 1$ is equal to $1/4$ and the probability of stepping forward to level $m + 1$ is $3/4$. Also, standing on the root you move to level 1 with probability 1. Because of this we can compute the probability of being at the origin after n steps by considering the following biased random walk on $\mathbb{N}$



Because it is only possible to be at the origin after an even number of steps and one must take the same number of steps to the right that to the left we have that

$$\begin{aligned}\sum_{n=0}^{\infty} u_n &= \sum_{n=0}^{\infty} u_{2n} \\ &= \sum_{n=0}^{\infty} \binom{2n}{n} \frac{3}{16}^{n} \\ &= 2\end{aligned}$$

Because the series converges, the random walk is transient. This means that with probability 1 every random walk will return to the origin only a finite number of times. We can generalize this assertion and say that the random walk will only visit any point a finite number of times. Because the ball of any given radius of the Cayley Graph of F_2 has finitely many points, with probability 1 a given random walk will eventually be arbitrarily far away from the origin.

1.3 Poisson Boundary

We will call an outcome of the random walk on $\mathbb{F}_2$ after taking an infinite number of steps, a trajectory. Let's consider Ω to be the set of all the possible trajectories of the random walk on $\mathbb{F}_2$. We say that two trajectories $\{x_n\}$ and $\{y_n\}$ in Ω have the same tail if there exist certain $t_1, t_2 \in \mathbb{N}$ such that $x_{t_1+n} = y_{t_2+n}$, $\forall n \in \mathbb{N}$. This defines an equivalence relation $\sim$ in Ω.

We can endow Ω with the probability measure $\mathbb{P} = \mu^{*\mathbb{N}}$. The Poisson Boundary is then the probability space defined by the quotient of $(\Omega, \mathbb{P})$ by the equivalence relation $\sim$. A more formal definition of the Poisson boundary and why it is in fact a probability space can be found in section 4 of [3].

2 Group Actions and Orbit Spaces

Recall the definition of a group action:

Definition 8. Let G be a group and X be a set. A *left action* of G on X is a map

$$G \times X \to X, \quad (g, x) \to g.x$$

such that for all $g_1, g_2 \in G$ and $x \in X$ we have

- $(g_1 g_2).x = g_1.(g_2.x)$
- $id.x = x$

Equivalently, we may say that an action of a group G on a set X is a homomorphism

$$G \to S_X$$

where S_X is the symmetric group on X. To be more precise, this is a *left action* of G on X. Given an element $x \in X$, we may define it's *orbit* as $orb_G(x) = \{g.x | g \in G\}$. The orbits induce an equivalence relation $x \sim y$ if and only if $x \in orb_G(y)$, and thus we may define the *quotient set* X/G as the set of orbits.

Naturally, when studying the action of groups on topological spaces, we should require our maps to be continuous. Whence, we define a *continuous left action* of a discrete group G on a topological space X as a group action whose action map is continuous. Equivalently, we may take a left action of G on X to be a group homomorphism

$$G \to Aut(X)$$

where $Aut(X)$ is the group of self homeomorphisms of X.

Unless otherwise stated, all groups will be discrete, all group actions are continuous, and we will denote an action by $G \circlearrowright X$.

A natural and important question now presents itself: what properties of a topological space are preserved when passing to the quotient? For example, if Γ is a discrete group acting on a manifold M, what are necessary and sufficient conditions under which M/Γ is itself a manifold?

Definition 9. Let G act on a topological space X.

- $G \circlearrowright X$ is *free* if for all $x \in X$, $g.x = x$ if and only if $g = id$.
- $G \circlearrowright X$ is *properly discontinuous* if for each $x \in X$, there is a neighborhood $x \in U_x \subset X$ such that $g.U_x \cap U_x = \emptyset$ for all $g \neq id \in G$.

The motivation for these definitions is immediate - given a free, properly discontinuous action on a Hausdorff space X, we may find a neighborhood about $x \in U \subset X$ such that the G-translates of U are all disjoint!

3 The Fundamental Group and Covering Spaces:

We briefly review the construction of the fundamental group, and describe the basic theory of covering spaces. We then proceed to describe the deck group of a covering space and the correspondence between the deck group of a universal cover and the fundamental group of the base space. The section is concluded by considering actions of the fundamental group on the universal covering space.

3.1 The Fundamental Group

Unless otherwise specified, all spaces are topological and all maps of spaces are continuous.

Definition 10. Let $f, g : X \to Y$ be maps. We say f and g are **homotopic** if there exists a map $h : X \times I \to Y$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$. The map h is called a **homotopy**. Given paths $\gamma, \alpha : I \to X$, we say γ is **path homotopic** to α, written $\gamma \simeq \alpha$, if there exists a homotopy $h : I \times I \to X$ such that $h(0, t) = \alpha(0) = \gamma(0)$ and $h(1, t) = \alpha(1) = \gamma(1)$.

Given two paths $\gamma, \alpha : I \to X$ such that $\gamma(0) = \alpha(1)$, the **concatenation** of α and γ is defined to be

$$\gamma \star \alpha(t) = \begin{cases} \alpha(2t) & t \in [0, \frac{1}{2}] \\ \gamma(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

One quickly verifies that the relation of path homotopy is an equivalence relation, and $[\gamma]$ will denote the equivalence class of γ. We now show that $\star$ descends to equivalence classes:

Proposition 3.1. *The operation* $[\gamma] \star [\alpha] := [\gamma \star \alpha]$ *is well defined.*

Proof. Let $\gamma \simeq_{h_1} \gamma'$ and $\alpha \simeq_{h_2} \alpha'$. It is immediately verified that

$$h(s,t) = \begin{cases} h_2(2s, t) & s \in [0, \frac{1}{2}] \\ h_1(2s-1, t) & s \in [\frac{1}{2}, 1] \end{cases}$$

is a homotopy from $\gamma \star \alpha$ to $\gamma' \star \alpha'$.

□

Let $Top_\star$ denote the category of pointed topological spaces and Grp be the category of groups. Given two spaces (X, x) and (Y, y), we denote by $[(X, x), (Y, y)]$ the homotopy classes of basepoint preserving maps from $(X, x) \to (Y, y)$.

Definition 11. Let $(X, x) \in Top_\star$. The **fundamental group** associated to (X, x) is defined as

$$\pi_1(X, x) = [(S^1, pt), (X, x)].$$

Furthermore, given $f : (X, x) \to (Y, y)$, the **induced homomorphism** $f_\star : \pi_1(X, x) \to \pi_1(Y, y)$ is defined as

$$f_\star([\gamma]) = [f \circ \gamma].$$

Our language in the previous definition was very suggestive, but the following proposition justifies the names in the previous definition.

Proposition 3.2. *The fundamental group* $\pi_1 : Top_\star \to Grp$ *defines a covariant functor.*

We only sketch the proof.

Proof. Concatenation of paths provides the binary operation. The identity in $\pi_1(X, x)$ is given by the constant loop c_x at the basepoint, and the inverse of $[\gamma(t)]$ is given by $[\gamma(1-t)]$. The group axioms now simply amount to writing down the appropriate homotopies, which we omit.

The induced map $f_\star$ is now obviously a homomorphism, $id_\star$ is the identity homomorphism, and $f_\star \circ g_\star = (f \circ g)_\star$ as desired.

□

Typically we will restrict our attention to path connected spaces. If X is path connected, then given $x, y \in X$ and a path $\alpha : x \to y$, we obtain an isomorphism $\pi_1(X, x) \to \pi_1(X, y)$ by $[\gamma] \to [\alpha \star \gamma \star \alpha^{-1}]$. Note that this isomorphism is not generally canonical (unless $\pi_1(X, x)$ is abelian), yet it allows us to ignore the basepoint when working with path connected spaces. If X and Y are path connected, then

$$\pi_1(X \times Y) \simeq \pi_1(X) \times \pi_1(Y)$$

(this follows quickly from the universal property of cartesian products). If, additionally to path connected, X and Y are locally contractible (e.g. real or complex manifolds), then

$$\pi_1(X \vee Y) \simeq \pi_1(X) * \pi_1(Y)$$

where $*$ denotes the free product of groups. We omit the justification of these useful identities.

3.2 Covering Spaces

Covering spaces are an important tool by which we can better understand fundamental groups.

Definition 12. A map $p : E \to B$ is said to be a **covering (covering space)** if p is surjective and for each $x \in B$, there exists an open neighborhood $x \in U \subset B$, such that $p^{-1}(U) = \sqcup_\alpha V_\alpha$ where $p\big|_{V_\alpha}$ is a homeomorphism onto U. The set U is called a **fundamental neighborhood** of x.

The following theorem provides a strong connection between fundamental groups and covering spaces.

Theorem 3.1. *Let $p : E \to B$ be a covering space, fix $b \in B$, and fix $e \in p^{-1}(b)$.*

- *Let $\gamma : I \to B$ be a path with $\gamma(0) = b$. Then there exists a unique path $\hat{\gamma} : I \to E$ such that $\hat{\gamma}(0) = e$ and $p \circ \hat{\gamma} = \gamma$. The path $\hat{\gamma}$ is called the* **lifted path** *of γ.*

- *Let $\gamma \simeq \alpha$ be equivalent paths in B. Then $\hat{\gamma} \simeq \hat{\alpha}$.*

Proof. Note that $im(\gamma) \subset B$ is compact. Cover $im(\gamma)$ by fundamental neighborhoods and extract a finite subcover

$$U_1, ..., U_n.$$

Partition I, $0 = t_1 < ... < t_m = 1$ such that $\gamma([t_i, t_{i+1}]) \subset U_j$ for some j. Define $\hat{\gamma}(0) = e$ and use the homeomorphism p^{-1} on the fundamental neighborhood containing $\gamma([t_1, t_2])$ to extend $\hat{\gamma}(0)$ to $[t_1, t_2]$ such that $p \circ \hat{\gamma} = \gamma\big|_{[t_1,t_2]}$. Proceeding inductively establishes the first claim.

Similarly, we may prove the second claim by subdividing the domain of the homotopy from $\gamma \to \alpha$ into subsquares, each of which is mapped into a fundamental neighborhood. We then lift each subsquare to E to obtain a homotopy from $\hat{\gamma} \to \hat{\alpha}$.

□

We now provide an example to illustrate the power of the last theorem.

Example: $\pi_1(S^1) \simeq \mathbb{Z}$

Consider S^1 as a subset of $\mathbb{C}$. Note that $p : \mathbb{R} \to S^1$ defined by $p(x) = e^{2\pi i x}$ is a covering map. Define $\varphi : \mathbb{Z} \to \pi_1(S^1, 1)$ via $\varphi(n) = [f_n]$ where $f_n(x) = e^{2\pi i n x}$. It is quickly verified that φ is a homomorphism. We will show that it is an isomorphism.

Define $\psi : \pi_1(S^1, 1) \to \mathbb{Z}$ by $\psi([\gamma]) = \hat{\gamma}(1)$ where $\hat{\gamma}$ is based at 0 in $\mathbb{R}$. This indeed defines an integer, for $p^{-1}(1) = \mathbb{Z}$. We must check that it is well defined, but this follows immediately from the second claim in Theorem 1.1. Furthermore, $\psi \circ \varphi = id_{\mathbb{Z}}$ and thus it suffices to check injectivity of ψ. If $\psi([\gamma]) = \psi([\alpha])$, then $\hat{\gamma}(1) = \hat{\alpha}(1)$ and thus $\hat{\alpha}^{-1} \star \hat{\gamma} \in \pi_1(\mathbb{R}, 0)$. But $\mathbb{R}$ is contractible, so $\hat{\alpha}^{-1} \star \hat{\gamma} \simeq c_0$. Thus

$$\alpha^{-1} \star \gamma \simeq c_1 \quad \implies \quad \gamma \simeq \alpha$$

as desired.

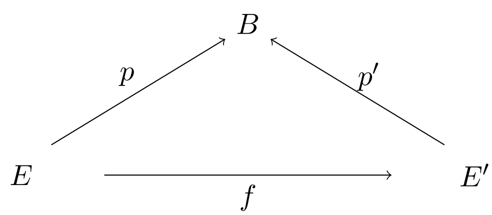
Definition 13. Let $p : E \to B$ be a covering space. If E is simply connected, we say that $p : E \to B$ is a **universal cover**.

We now further explore the relationship between covering spaces and the fundamental group by introducing maps of covering spaces.

3.3 Deck Transformations

Unless otherwise specified, all spaces will be assumed to be connected and locally path connected. In particular, note that these two conditions imply path connectedness of our spaces.

Definition 14. Let $p : E \to B$ and $p' : E' \to B$ be covering spaces. A *morphism of covering spaces* is a commutative diagram:



A *deck transformation* is an automorphism of covering spaces.

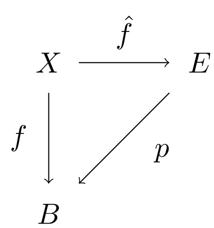
Given a covering space $p : E \to B$, the set of deck transformations $G(E)$ forms a group under composition. The main result of this section is the following theorem:

Theorem 3.2. *Let $p : (E, e) \to (B, b)$ be a covering space and $H = p_\star(\pi_1(E, e)) \subset \pi_1(B, b)$. Then*

$$G(E) \simeq N(H)/H.$$

In order to prove this, we will need a vast generalization of Theorem 1.1. The following theorem gives this generalization and is very useful, not only for proving the previous theorem, but also for classifying covering spaces. The theorem states that the fundamental group gives the only obstruction to the existence of a certain map - a deep and interesting result in it's own right.

Theorem 3.3. *Let $p : (E, e) \to (B, b)$ be a covering and $f : (X, x) \to (B, b)$. Then $f_\star(\pi_1(X, x) \subset p_\star(\pi_1(E_e))$ if and only if there exists a unique map $\hat{f} : (X, x) \to (E, e)$ such that the following diagram commutes:*



Proof. ($\Longrightarrow$) Let $[\gamma] \in \pi_1(X, x)$. Then $\hat{f}_\star([\gamma]) \in \pi_1(E, e)$ and $p_\star \circ \hat{f}_\star([\gamma]) = f_\star([\gamma])$.

($\Longleftarrow$) We begin by defining our the map $\hat{f}$. Given $a \in X$, let $\alpha : I \to X$ be a path from $x \to a$. We define

$$\hat{f}(a) = \overline{f \circ \alpha}(1)$$

where $\overline{f \circ \alpha}$ is the unique lift of $f \circ \alpha$ to E based at e. The condition that $f_\star(\pi_1(X, x)) \subset p_\star(\pi_1(E, e))$ insures that $\hat{f}$ is independent of our choice of α, so we need only show that $\hat{f}$ is continuous.

We may take as a basis for the topology on E, $\mathscr{B} = \{U_\alpha | p\big|_{U_\alpha}$ is a homeomorphism onto it's image$\}$. It suffices to show $\hat{f}^{pre}(U_\alpha)$ is open in X for $U_\alpha \in \mathscr{B}$, so fix $U_\alpha \in \mathscr{B}$. We know $f^{-1}(p(U_\alpha))$ is open in X. Pick $z \in U_\alpha \cap im\hat{f} \neq \emptyset$ (if it is empty the claim follows immediately), and pick $z \in V \subset f^{-1}(p(U_\alpha))$ which is open and path connected (recall all spaces are locally path connected). By path connectedness, we obtain $p^{-1} \circ f = \hat{f}$ which implies that $V \subset \hat{f}^{-1}(U_\alpha)$. This implies $\hat{f}^{-1}(U_\alpha)$ is indeed open in X which establishes the theorem. □

The following immediate corollary will be useful for the purposes of proving Theorem 1.2:

Corollary 3.1. *Let $e, e' \in p^{-1}(b)$. There exists a unique $f \in G(E)$ such that $f(e) = e'$ if and only if $p_\star(\pi_1(E, e)) = p_\star(\pi_1(E, e'))$.*

The final ingredient needed to implement a proof of Theorem 1.2 is the following:

Lemma 3.1. *Let $H = p_\star(\pi_1(E, e))$ and $e' \in p^{-1}(b)$. Let $\gamma : I \to E$ be a path from $e \to e'$. Then $[p \circ \gamma] \in N(H)$ if and only if $p_\star(\pi_1(E, e)) = p_\star(\pi_1(E, e'))$.*

Proof. Let $c_\gamma : \pi_1(E, e) \to \pi_1(E, e')$ denote conjugation by $[\gamma]$ and $c_{p\circ\gamma}$ be defined analogously. If $[p \circ \gamma] \in N(H)$ then we observe that $c_{p\circ\gamma} = p_\star \circ c_\gamma : H \to p_\star(\pi_1(E, e'))$ is an isomorphism. If $p_\star(\pi_1(E, e')) = H$, then the reverse implication is immediate.

For the forward implication, we merely observe that $c_{p\circ\gamma}$ is an automorphism of H. □

Proof. (of Theorem 1.2) From the previous Lemma and corollary of Theorem 1.3 we have the following chain of equivalences:

$$[\gamma] \in N(H) \iff p_\star(\pi_1(E, e)) = p_\star(\pi_1(E, \hat{\gamma}(1))) \iff \exists! f \in G(E) \ni f(e) = \hat{\gamma}(1)$$

So we immediately obtain a surjective map $\Phi : N(H) \to G(E)$ by $\Phi([\gamma]) = f_\gamma$. Evidently $\Phi([\gamma]) = id$ if and only if $[\gamma] \in H$. Thus it suffices to show that Φ is a homomorphism. This follows quickly for given $[\gamma], [\alpha] \in N(H)$ we have

$$\overline{\gamma \star \alpha}(1) = \overline{\gamma} \star \hat{\alpha}(1) = \overline{\gamma}(1)$$

where $\overline{\gamma}$ is lifted to $\hat{\alpha}(1)$ and thus

$$\overline{\gamma}(1) = f_\gamma \circ f_\alpha$$

as desired. □

Thus for a universal cover $p : E \to B$, we have a natural way in which to define an action $\pi_1(B) \circlearrowright E$ - namely via deck transformations. This action clearly depends on the projection p, which is by no means unique. The choice of projection amounts to choice of faithful representation of $\pi_1(B)$ into $Homeo(E)$

4 The Word Length Ratio

Definition 15. Let $C(G)$ be the Cayley graph of a finitely generated group G and let H be a finitely generated subgroup of G. We will define a new graph $\hat{C}(G)$ called the *coned-off Cayley Graph* of G as follows:

1. Start with $C(G)$, for each coset gH add a new vertex $v(gH)$ to $C(G)$.
2. Add an edge $e(gh)$ of length $1/2$ from each element gh of gH to the vertex $v(gH)$.

The shortest path between two points endows this graph with a metric.

Definition 16. Since $C(G) \subset \hat{C}(G)$ this gives us a new metric in G called *relative metric.* In this new metric all the distinct elements of the same coset are now at distance 1 form each other. We will denote the relative metric between two points $g_1, g_2 \in G$ by $d_{rel}(g_1, g_2)$. From now on to avoid any confusion we will denote by $d_G(g_1, g_2)$ the usual word length metric in G.

Definition 17. A group G is hyperbolic relative to H if the coned-off Cayley graph of G with respect to H is a negatively curved metric space.

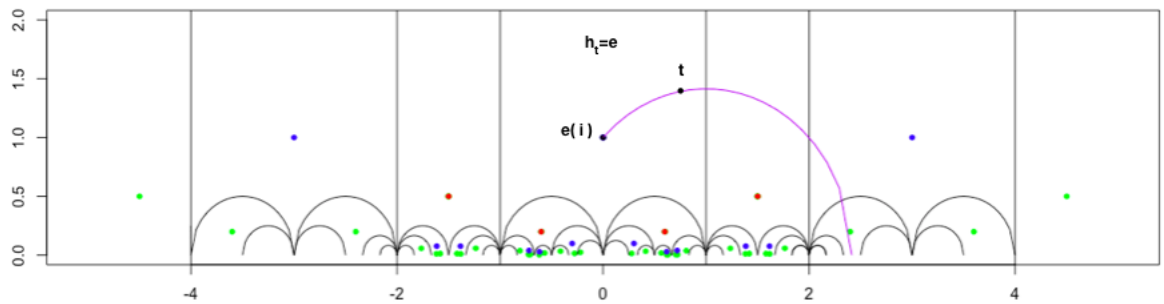
Definition 18. Let G be a finitely generated group hyperbolic relative to H. Consider the Cayley Graph of G to be embedded in a metric space (M, d) and a random walk on G given by the probability measure μ. Let γ be a geodesic ray from the base point x_0 and let γ_t be a point at distance t from the base point. For each time t consider h_t to be an element in G such that $h_t x_0$ is an element in the orbit x_0 that is closest to γ_t under d. We then define the quantity

$$\rho(\gamma) := \lim_{t\to\infty} \frac{d_G(e, h_t)}{d_{rel}(e, h_t)}.$$

This quantity will be called the **word length ratio**.

Theorem 4.1 (Gadre-Maher-Tiozzo)**.** *Let G be a discrete group that acts on $\mathbb{H}$ by isometries such that $\mathbb{H}/G$ has finite hyperbolic area and at least one cusp. Let ν be a harmonic measure on the boundary $\partial\mathbb{H}$ determined by the probability distribution μ whose support is finite and generates G. Take H to be a subgroup of G that fixes the cusp. Then there is a constant $c > 0$ such that $\rho(\gamma) = c$ for ν-almost all geodesic rays. [4]*

We will be interested in studying the word length ratio of geodesics in the hyperbolic plane and taking $G = \mathbb{F}_2$ to be the fundamental group of the punctured torus. For this case, G acts by isomerties and $\mathbb{H}/G$ has finite hyperbolic area. We will consider $H =< a^{-1}b^{-1}ab >$ since this subgroup of G fixes the cusp and G is hyperbolic relative to H (see [5], section 3).



Notice that, multiplying by an element in H corresponds to going around the cusp. So, a way of interpreting c is by thinking of it as an average of how many times ν-almost all geodesic rays wrap around the cusp.

5 Comparing the relative metric with the word metric

Remark 3. For any two elements $g_1, g_2 \in G$ we have that

$$d_{rel}(g_1, g_2) \leq d_G(g_1, g_2).$$

This is given by the fact that the Cayley graph of G is a subgraph of the relative Cayley graph of G.

Definition 19. Consider the following elements in $\mathbb{F}_2$

$$T_1 = a^{-1}b^{-1}a$$
$$T_2 = b^{-1}a^{-1}b$$
$$B = a^{-1}b^{-1}ab$$
$$B^{-1} = b^{-1}a^{-1}ba$$

we will call these elements **blocks**, more precisely T_1 and T_2 will be called 3-blocks and B and B^{-1} will be called 4-blocks.

Remark 4.

$$d_{rel}(B, e) = d_{rel}(B^{-1}, e) = 1$$
$$d_{rel}(T_1, e) = d_{rel}(T_2, e) = 2$$

Proof. The first equalities are clear since the identity, B and B^{-1} all belong to the same coset. For the second equality notice that B is at distance 1 from T_1 and B^{-1} is at distance 1 form T_2. Since T_1 and T_2 are at distance 1 from the identity there exist a path of length 2 in the relative Cayley graph between T_1 and e and between T_2 and e. This implies that $d_{rel}(T_1, e) \leq 2$ and $d_{rel}(T_2, e) \leq 2$. Since T_1 and T_2 are not in the same coset as e and $d_G(T_1, e) = d_G(T_2, e) = 3$ we and conclude that the second equality holds. □

Definition 20. An element $g \in \mathbb{F}_2$ will be said to be **clean** if its reduced word representation does not contain any blocks as subwords.

Lemma 5.1. *If $g \in \mathbb{F}_2$ is clean then $d_{rel}(g, e) = d_G(g, e)$.*

Proof. First, notice that all the edges in the conned-off Cayley Graph have length 1/2 or 1. Also, the coset vertices are not connected to each other. Then, for any $g \in G$, the elements of the group that are at distance 1 from g are

$$\mathbb{B}_1(g) = \{g \cdot a, \quad g \cdot b, \quad g \cdot a^{-1}, \quad g \cdot b^{-1}\} \cup gH$$

.

To find $d_{rel}(g, e)$ we must find the smallest $n \in \mathbb{N}$ so that there exists a path

$$g = g_0 \longrightarrow g_1 \longrightarrow g_2 \longrightarrow \cdots \longrightarrow g_{n-1} \longrightarrow g_n = e$$

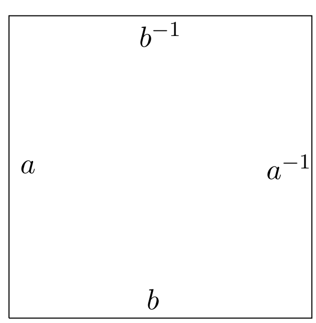
where $g_i \in \mathbb{B}_1(g_{i-1})$ for $i = 1, ..., n$. Notice that this corresponds to finding an optimal sequence of n of the following actions that leads to eliminating all of the letters in g:

1. Adding a letter at the end of the reduced word representation of g_i.
2. Deleting a letter at the end of the reduced word representation of g_i.
3. Deleting a sequence of letters at the end of the reduced word representation of g_i that corresponds to a power of $a^{-1}b^{-1}ab$. That is, deleting B^k or B^{-k}.

Suppose g is clean, then g doesn't end in a 4-block so we can't use action 3 directly. If we try adding letters (action 1) to the end of g we will find that because g has no 3-blocks we must add at least two letters before being able to apply action 3. So, to delete the last two letters this way we must make at least two actions. Because we can apply action 2 twice we know that we can always delete the last two letters with exactly two actions. Proceeding this way we find that if if $d_G(g, e) = n$ we can delete all the letters in g with n actions and no less. We can conclude that if g is clean then $d_G(g, e) = d_{rel}(g, e)$. □

6 Cutting Sequences

As a motivating example, consider the torus T^2, whose universal covering space is $\mathbb{R}^2$. Note that $\pi_1(T^2) = \pi_1(S^1 \times S^1) \simeq \pi_1(S^1) \times \pi_1(S^1) \simeq \mathbb{Z} \times \mathbb{Z}$, and thus we define the representation $\pi_1(T^2) \to Isom(\mathbb{R}^2)$ via $(0, 1) \to [b(x) = x + (0, 1)]$, and $(1, 0) \to [a(x) = x + (1, 0)]$. The tessellation in $\mathbb{R}^2$ induced by the generators a, b is merely a uniform grid with integer valued vertices. Each square in the grid may be viewed as a pasting diagram (indeed, this defines the covering map induced by our chosen representation), and we may label each edge with the generator which identifies it:



Now, a geodesic γ on T^2 lifts to a straight line L in $\mathbb{R}^2$. Endowing the geodesic with a starting point and an orientation, we may encode the geodesic by a sequence of generators according to the order in which it crosses the labelled edges. The resulting sequence in $\{a, b, a^{-1}, b^{-1}\}$ is called the **cutting sequence of** γ. A simple question arises:

What sequences in $\{a, b, a^{-1}, b^{-1}\}$ arise as the cutting sequence of a geodesic?

Definition 21. A sequence of the form

$$ab^{n_1}ab^{n_2}ab^{n_3}... \quad n_i \in \{k, k+1\}$$

will be called **almost constant**, and k will be called it's **value**.

Note that given an almost constant sequence, we may view the variable substitution $a = ab^k$ and $b = b$ as a linear isomorphism of $T : \mathbb{R}^2 \to \mathbb{R}^2$. The resulting sequence is the **derived sequence** Since cutting sequences of lines are necessarily almost constant, observe that the cutting sequence of the line $T^{-1}(L)$ is precisely the derived sequence. This derivation process depends only on the sequence being almost constant, and thus we define a sequence which admits arbitrarily many derivations **characteristic**. We arrive at our first result on cutting sequences:

Theorem 6.1. *The cutting sequence of a line L is characteristic, and the successive values $n_0, n_1, ...$ of it's derived sequence satisfy*

$$slope(L) = n_0 + \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + ...}}}$$

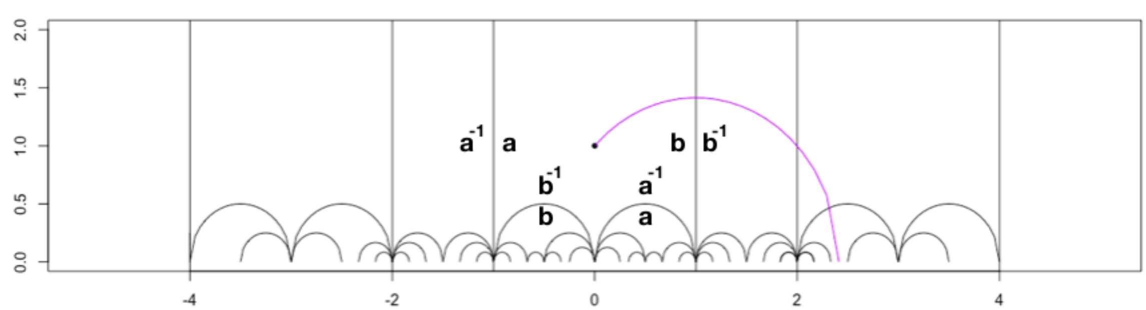
We will omit the justification of this theorem, as well further theorems in this section. The interested reader may refer to [7] for further details.

Perhaps somewhat surprisingly, we have encoded geodesics in the torus in such a way as to be able to recover certain arithmetic data attached to the geodesic. All that was necessary was a representation of the fundamental group into the isometries of the universal cover, and a labelling of the induced tessellation. Indeed, nothing prevents us from playing the same game with the punctured torus.

The universal cover of $T^2 - pt$ is $\mathbb{H}$ - the hyperbolic plane, and the fundamental group is

$$\pi_1(T^2 - pt) \simeq \pi_1(S^1 \vee S^1) \simeq \mathbb{Z} \star \mathbb{Z} = F_2$$

the free group on two generators (the first isomorphism above is given by a deformation retraction - not covered in our treatment of the fundamental group. See [6]). We may choose the representation of $\pi_1(T^2 - pt)$ to be generated by $a(z) = \frac{z+1}{z+2}$ and $b(z) = \frac{z-1}{-z+2}$, generating the tessellation and labelling as seen below.



We may play the same game in this new context, endowing geodesics with cutting sequences. The following two results about cutting sequences in this context will be relied upon in the following sections.

Proposition 6.1. *A geodesic on $T^2 - pt$ is simple and closed if and only if its cutting sequence is periodic and characteristic.*

By the cutting sequence of a geodesic on $T^2 - pt$, we simply mean the cutting sequence of any of its lifts (the cutting sequence is independent of our choice of lift). By characteristic, we mean exactly what was meant in the Euclidean case (in particular, the cutting sequence of a simple closed curve only contains two letters).

Proposition 6.2. *Given any sequence $\{x_n\}$ in $\{a, b, a^{-1}, b^{-1}\}$ such that a letter is never followed by its inverse, and the sequence is not $aba^{-1}b^{-1}aba^{-1}b^{-1}...$, there exists a geodesic whose cutting sequence is $\{x_n\}$.*

7 Translating Between h_t and Cutting Sequences

The main goal of this section is to reach the following conjecture:

Conjecture 1: Let γ be a geodesic in $\mathbb{H}$ such that $\lim_{t\to\infty} = \theta \in \mathbb{R} \setminus \mathbb{Q}$. Then $\lim_{t\to\infty} h_t$ is the cutting sequence of γ.

Notice that this would imply that the word length ratio defined by h_t would correspond to the the word length ratio given by the cutting sequence.

Let G be a discrete group acting on $\mathbb{H}$ freely and properly discontinuously by isometries. We will primarily be interested in the action of the free group on two generators, though many of the following statements hold in higher generality.

Definition 22. Given $x \in \mathbb{H}$, the **Dirichlet domain** of x with respect to the action $G \circlearrowright \mathbb{H}$ is defined as

$$\mathscr{D}(x) = \{z \in \mathbb{H} | d_{\mathbb{H}}(z, x) \leq d_{\mathbb{H}}(g.z, x) \forall g \in G\}.$$

A **fundamental domain** of the action $G \circlearrowright \mathbb{H}$ is a closed subset $D \subset \mathbb{H}$ such that $\mathbb{H} = \bigcup_{g\in G} g.D$ and for all $g, g' \in G$, $int(g.D) \cap int(g'.D) = \emptyset$.

Upon embedding the Cayley graph Γ_G into $\mathbb{H}$, the Dirichlet domains of the elements of Γ_G give a natural connection between h_t and γ - namely if $\gamma(t) \in \mathscr{D}(a)$ for $a \in \Gamma_G$, then $h_t = a$. To fully exploit this property we must expand our theory of Dirichlet domains, showing that in fact, the Dirichlet domain of e is a fundamental domain for the G-action.

Lemma 7.1. *The $G-$translates of a Dirichlet domain are the Dirichlet domains of the G-translates. That is, given $g \in G$ and $x \in \mathbb{H}$, we have $g.\mathscr{D}(x) = \mathscr{D}(g.x)$.*

Proof. This follows immediately from the property that G acts on $\mathbb{H}$ by isometries. Fix $g \in G$ and $x \in \mathbb{H}$, then

$$g.\mathscr{D}(x) = \{g.z | d(x, z) \leq d(x, a.z) \forall a \in G\} = \{z | d(x, g^{-1}.z) \leq d(x, a.(g^{-1}.z)) \forall a \in G\}$$

$$= \{z | d(g.x, z) \leq d(g.x, a.x) \forall a \in G\} = \mathscr{D}(g.x)$$

as desired.

□

In other words, Dirichlet domains are preserved under the G-action. We now proceed to show that the G-translates of a single Dirichlet domain cover the plane.

Lemma 7.2. *Given $x, a \in \mathbb{H}$, there exists $y \in orb_G(a)$ such that $x \in \mathscr{D}(y)$.*

Proof. Suppose for all $y \in orb_G(a)$, $x \notin \mathscr{D}(y)$. Let $\delta = inf\{d(x, y) | y \in orb_G(a)\}$, and consider the closed ball

$$V = \overline{B(x, \delta + 1)}.$$

Then V contains infinitely many orbit points, yet is compact - contradicting proper discontinuity of the G-action.

□

Lemma 7.3. *Let $x \in \mathbb{H}$. Then for all $g \in G$, $int(\mathscr{D}(x)) \cap int(\mathscr{D}(g.x)) = \emptyset$.*

Proof. We begin by proving for $A = \{z \in \mathbb{H} | d(z, x) < d(g.z, x) \forall g \in G\}$, that $A = int(\mathscr{D}(x))$. Clearly we have $A \subset int(\mathscr{D}(x))$. If $a \in \mathscr{D} \setminus A$, then $d(a, x) = d(g.a, x)$ for some $g \in G$. Thus $d(a, x) = d(g.z, x) = d(g.z, g.x)$ which implies for all $\epsilon > 0$, there exists an $x \in B(g.a, \epsilon)$ for which $d(a, g.x) < d(a, x)$ since there are distinct geodesics from $x \to g.a$ and $g.x \to g.x$.

Observe since the $G-$action is free, no two Dirichlet domains coincide. Now, if $y \in int(\mathscr{D}(x)) \cap int(\mathscr{D}(g.x))$, then $d(x, y) < d(x, g.y)$ and $d(g.x, y) < d(g.x, g.y)$. Whence

$$d(x, y) > d(g.x, y) \text{ which implies } d(x, g^{-1}.y) < d(x, y)$$

contrary to supposition.

□

Combining the three lemmas, we arrive at the following conclusion:

Proposition 7.1. *Fix $x \in \mathbb{H}$. Then $\mathscr{D}(x)$ is a fundamental domain for the group action $G \circlearrowright \mathbb{H}$.*

We now return to the case of the punctured torus, with the specified action of it's fundamental group on $\mathbb{H}$, and $H =< a^{-1}b^{-1}ab >$. Ultimately, our goal is to classify the geodesics γ for which we may substitute the cutting sequence x_n of γ for h_t in the ratio $\frac{d_G(1,h_t)}{d_{rel}(1,h_t)}$ and preserve $\rho(\gamma)$. Evidently, if the endpoint of γ lies in $\mathbb{Q}$, this will not be possible - but if γ limits to an irrational the question is far less clear. We thus turn our attention towards reduced sequences of generators.

Given a sequence of generators x_n, let $r(x_n) = \frac{d_G(1,\prod_{i=1}^n x_i)}{d_{rel}(1,\prod_{i=1}^n x_i)}$. Obviously, we may find reduced sequences x_n for which $r(x_n)$ diverges to infinity (just list $a^{-1}b^{-1}ab$ ad infinitum). When does $r(x_n)$ converge? We would like to say that given a reduced sequence x_n, if $r(x_n)$ is bounded then it is a Cauchy sequence. This is not the case - one may construct a bounded sequence which doesn't converge. But our intuition about the successive terms in the sequence $r(x_n)$ 'bunching up' isn't entirely wrong - indeed, the next best thing happens:

Lemma 7.4. *Suppose there exists $M > 1$ such that $r(x_n) = \frac{n}{b_n} \leq M$ for all n. Then for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for each $n \geq N$,*

$$|r(x_{n+1}) - r(x_n)| < \epsilon.$$

Proof. Let $\epsilon > 0$ and pick N such that for each $n \geq N$, $\frac{2M}{b_n} < \frac{\epsilon}{2}$. Then we have

$$\begin{aligned} |\frac{n}{b_n} - \frac{n+1}{b_{n+1}}| &= |\frac{(b_{n+1} - b_n)n}{b_n b_{n+1}} - \frac{1}{b_{n+1}}| \\ &\leq |\frac{(b_{n+1} - b_n)M}{b_{n+1}}| + \frac{\epsilon}{2} < \epsilon \end{aligned}$$

□

Corollary 7.1. *Let x_n be a reduced sequence of generators and x_{n_k} be a subsequence such that $\lim_{k\to\infty} r(x_{n_k}) = \alpha \in \mathbb{R} \cup \{\infty\}$. If there exists $M \in \mathbb{R}$ such that $n_{k+1} - n_k < M$ for all k, then $\lim_{n\to\infty} r(x_n) = \alpha$.*

The following corollary of Lemma 7.1 will allow this substitution to take place for a large class of geodesics.

Corollary 7.2. *Let x_n be a reduced sequence of generators and $M > 0$. For each n, let $x_n^\star$ be defined as x_k for the first n letters and any sequence of length M thereafter. Then*

$$\lim_{n\to\infty} r(x_n^\star) = \lim_{n\to\infty} r(x_n).$$

Let $G \circlearrowright \mathbb{H}$ freely, properly discontinuously, by isometries where G is a dicrete group. Given an embedding $C(G) \xrightarrow{f} \mathbb{H}$, we know that $\mathscr{D}(f(id))$ is a fundamental domain for the action.

Lemma 7.5. *Let $\gamma : [0, \infty] \to \mathbb{H}$ be a geodesic ray and partition $[0, \infty]$ by*

$$0 = t_1 < t_2 < ...$$

such that $h_t = h_{t_i}$ for all $t \in (t_{i-1}, t_i]$. Then there exists $M > 0$ such that

$$|l_s(h_{t_{i-1}}) - l_s(h_{t_i})| < M$$

Proof. Note that $\gamma(0) \in \mathscr{D}(h_0)$. Since $\mathscr{D}(h_0)$ is a fundamental domain, it shares it's boundary with finitely many Dirichlet domains of $g.h_0$. Let $M = max\{l_s(g.h_0) | \mathscr{D}(g.h_0) \cap \mathscr{D}(h_0) \neq \emptyset\}$. We thus see that $|l_s(h_0) - l_s(h_{t_1}| < M$. Since G acts by isometries, we see proceed inductively by mapping $h_0 \to h_{t_i}$.

□

Thus, using the approximation lemma, we see that if $\lim_{t\to\infty} h_t = \lim_{t\to\infty} \gamma(t) = \theta$, and $\theta \notin \mathscr{D}(g.h_0)$ for any $g \in G$, then $\rho(\gamma) = \lim_{t\to\infty} r(x_t)$ where x_t is the cutting sequence of $\gamma\big|_{[0,t]}$. In the case $G = \pi_1(T^2 - pt)$, these conditions are met precisely when θ is irrational - as we now show.

Proposition 7.2. *Given the embedding $\pi_1(T^2 - pt) \to \mathbb{H}$ defined by $id \to i$,*

$$\partial\mathscr{D}(i) \cap \partial\mathbb{H} = \infty.$$

Proof. It suffices to prove that $\mathscr{D}(i)$ contains no points whose imaginary part lies on the line $im(z) = \frac{1}{10}$. We spare the reader the details of the various computations.

Observe that $d(i, \frac{i}{10}) > d(\frac{i}{10}, -\frac{3}{10} + \frac{i}{10}) = d(\frac{i}{10}, a \circ b(i))$. Thus, by symmetry of the tessellation, we see that

$$\{z \in \mathbb{H} | im(z) = \frac{1}{10}, re(z) \in [-\frac{3}{5}, \frac{3}{5}]\} \cap \mathscr{D}(i) = \emptyset$$

Furthermore, one may verify that $d(i, \frac{2}{5} + \frac{i}{10}) > d(a(i), \frac{2}{5} + \frac{i}{10})$, from which we appeal to symmetry again to see that any point on $im(z) = \frac{1}{10}$ whose real part satisfies $re(z) \in [-\frac{4}{5}, \frac{4}{5}]$ also doesn't lie in $\mathscr{D}(i)$. But

$$d(i, \frac{4}{5} + \frac{i}{10}) > d(b^{-1}(i), \frac{4}{5} + \frac{i}{10})$$

from which the proposition follows.

□

Corollary 7.3. *For all $g \in \pi_1(T^2 - pt)$, $\partial\mathscr{D}(g.i) \cap \partial\mathbb{H} \in \mathbb{Q} \cup \infty$*

Proof. The orbit of ∞ under $\pi_1(T^2 - pt)$ is $\mathbb{Q}$.

□

Corollary 7.4. *Let γ be a geodesic in $\mathbb{H}$ with $\lim_{t\to\infty} \gamma(t) = \theta \in \mathbb{R}\backslash\mathbb{Q}$. Then $\lim_{t\to\infty} h_t = \theta$.*

The final ingredient needed to implement a proof of Conjecture 1 is the following conjecture.

Conjecture 2: The fundamental domain of the tessellation induced by the representation of $\pi_1(T^2 - pt)$ is covered by finitely many Dirichlet domains about the orbit of i, and vice versa.

8 Applications to $\rho(\gamma)$

Given Conjectures 1 and 2, we may deduce several more interesting results.

Proposition 8.1. *Let $\alpha \in \mathbb{R}$, $\alpha \geq 1$. Then there exists a geodesic γ on the punctured torus such that $\rho(\gamma) = \alpha$.*

Proof. We begin by stating two simple observations about a reduced sequence of generators x_n. First, if there exists N such that for all $n \geq N$, the x_n is the $(n \text{ mod} 4)^{th}$ letter of B, then $\lim_{n\to\infty} x_n = \infty$. Indeed, merely consider the subsequence $n_k = N + 4k$, for which $r(x_{n_k}) = \frac{i+4k}{j+1}$. We see that $\lim_{k\to\infty} r(x_{n_k}) = \infty$ and apply Corollary 7.1.

On the other hand, if there exists N such that for all $n \geq N$, $x_n = a$, then $\lim_{n\to\infty} x_n = 1$. Again, we simply pass to an appropriate subsequence and apply Corollary 7.1. We now construct a reduced sequence x_n for which $\lim_{n\to\infty} r(x_n) = \alpha$.

Step 1: Let $B = a^{-1}b^{-1}ab$ and set x_k equal to the $(k \text{ mod} 4)^{th}$ element of B until

$$r(x_n) > \alpha \geq r(x_{n-4}).$$

By the first observation above, this will always happen in finite time, after which we proceed to step 2.

Step 2: Set $x_k = b$ until

$$r(x_n) < \alpha \leq r(x_{n-1}).$$

Again, this is always possible by the preceding comments. We now repeat step 1.

Hence, we have a sequence of generators x_n such that $r(x_n) \leq \alpha + 1$ for all n and for which there are infinitely many indices n such that $r(x_n) < \alpha$ and infinitely many indices k such that $r(x_k) > \alpha$. Applying Lemma 7.1 establishes the claim.

□

Proposition 8.2. *If γ is a simple closed curve, then $\rho(\gamma) = 1$.*

Proof. The cutting sequence of a simple closed curve is clean (see the final results in the section on cutting sequences).

□

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