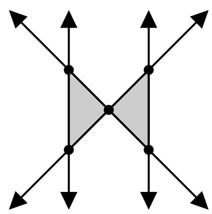
41ST UNIVERSITY OF MICHIGAN UNDERGRADUATE MATHEMATICS COMPETITION

1pm-4pm, April 6, 2024

Problem 1. Determine the value of the following limit or prove that it does not exist:

$$\lim_{n\to\infty} \frac{n + \sqrt{n} + \sqrt[3]{n} + \cdots + \sqrt[n]{n}}{n}.$$

Problem 2. Consider a configuration of $n \geq 2$ distinct lines in $\mathbb{R}^2$. Assume that no three lines meet in a single point, while there are $x \geq 1$ intersection points of pairs of lines. Suppose the lines cut the plane into b bounded regions (and some number of unbounded regions). For example, in the diagram below we have $n = 4$, $x = 5$, and $b = 2$. Prove that $b = x - n + 1$.



Problem 3. Suppose $f : \mathbb{Z} \setminus \{0\} \to \mathbb{Z} \setminus \{0\}$ is a function from the nonzero integers to the nonzero integers satisfying

$$f(a) - f(b) = f(c) - f(d) \quad \text{whenever} \quad \frac{a}{b} = \frac{c}{d}.$$

Prove that f is not surjective.

Problem 4. Call a quadrilateral *elegant* if at most one of its four sides has irrational length. Prove or disprove: any quadrilateral can be cut into finitely many pieces, each of which is an elegant quadrilateral.

Problem 5. Let $n \geq 3$ and let $P_1, \ldots, P_n$ be the vertices of a regular n-gon with sides of length 1. Define an n-by-n matrix $M(n)$ by letting $M(n)_{ij}$ be the distance between P_i and P_j as one walks around the perimeter of the n-gon (in whichever direction is shorter). Here are the first two such matrices as examples:

$$M(3) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad M(4) = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}.$$

What is the rank of $M(2024)$?

Problem 6. Place 2024 particles around a circle, equally spaced. For each particle, flip a coin to choose either clockwise or counterclockwise. At time $t = 0$, each particle starts moving in the chosen direction at a speed of one full revolution around the circle per unit time. Whenever two particles collide, they each reverse direction and continue moving at the same speed. What is the probability that each particle is in the same position at time $t = 506$ as at time $t = 0$? (The particles are not identical - you can tell the difference between them.)

Problem 7. For any function $g : \mathbb{R} \to \mathbb{R}$, let $\mathrm{Per}(g)$ be the set of *periods* of g, i.e. $\mathrm{Per}(g) = \{p \in \mathbb{R} \mid g(x + p) = g(x) \text{ for all } x \in \mathbb{R}\}$. Now suppose $f : \mathbb{R} \to \mathbb{R}$ is a C^∞ function (i.e. can be differentiated arbitrarily many times). Prove that the infinite sequence of periods of derivatives of f,

$$\mathrm{Per}(f), \mathrm{Per}(f'), \mathrm{Per}(f''), \mathrm{Per}(f^{(3)}), \ldots$$

must eventually stabilize to a single set $\mathrm{Per}(f^{(n)}) = \mathrm{Per}(f^{(n+1)}) = \mathrm{Per}(f^{(n+2)}) = \cdots$ (for some sufficiently large n depending on f).

Problem 8. Let Sym_6 be the group of permutations of $\{1, 2, 3, 4, 5, 6\}$. For each $1 \leq i \leq 6$, let G_i be the subgroup of Sym_6 consisting of the permutations σ with $\sigma(i) = i$. Suppose H is a subgroup of Sym_6 that has order 120 but is not equal to any of the G_i. Prove that for any i, the subgroup $H \cap G_i$ has order 20 and is nonabelian.

Problem 9. Let X be a nonempty finite topological space (i.e. it is a finite set with a topology). Recall that X is *connected* if it cannot be expressed as the disjoint union of two nonempty open subsets, and X is *path-connected* if given any $x, y \in X$ there exists a continuous function $f : [0, 1] \to X$ with $f(0) = x, f(1) = y$. Prove that X is connected if and only if X is path-connected.

Problem 10. Let $n \geq 2$ and let $x_1, \ldots, x_n$ be real numbers. For any subset A of $\{1, 2, \ldots, n\}$, let $|A|$ be the number of elements in A and let $x_A = \sum_{i \in A} x_i$ be the sum of the corresponding x_i. Prove the identity

$$\sum_{A \sqcup B = \{1,\ldots,n\}} x_A^{|A|-1} x_B^{|B|-1} = (n-1)(x_1 + \cdots + x_n)^{n-2},$$

where the sum runs over all ways of writing $\{1, \ldots, n\}$ as the disjoint union of two nonempty subsets. For example, when $n = 3$ this says $(x_1 + x_2) + (x_1 + x_3) + (x_2 + x_3) = 2(x_1 + x_2 + x_3)$.

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