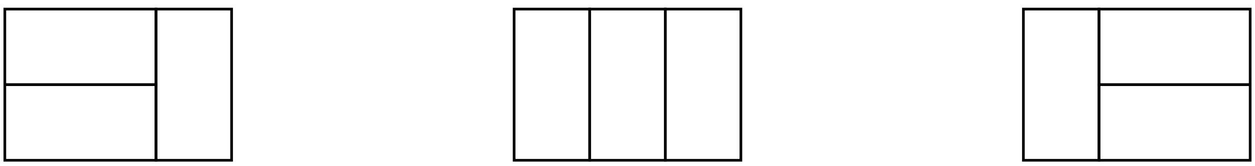
40TH UNIVERSITY OF MICHIGAN UNDERGRADUATE MATHEMATICS COMPETITION

1pm-4pm, April 1, 2023

Problem 1. Suppose that m and n are positive integers. Prove that there are an odd number of ways of filling a $(2m) \times (2^n - 1)$ rectangle with 1×2 dominoes. For example, there are 3 ways to fill a 2×3 rectangle.

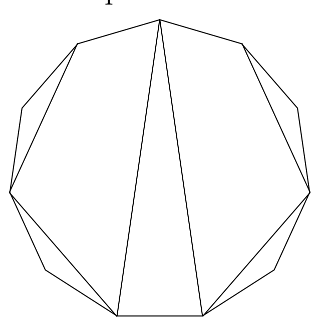


Problem 2. Find all continuous real-valued functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\int_a^b f(x)dx \leq f(b-a)$$

for all real numbers a, b.

Problem 3. Let P_n be a regular polygon with $n \geq 4$ sides. Consider ways of dissecting P_n into T triangles and Q quadrilaterals by drawing some internal diagonals connecting vertices of the polygon. Assume that these diagonals do not cross each other. Further assume that every quadrilateral shares at most one side with the original regular polygon P_n. Prove that $T \geq Q + 2$. (The diagram shows an example dissection with $T = 5, Q = 2$.)



Problem 4. Suppose that V is a real vector space with (infinite) basis e_n indexed by the integers $n \in \mathbb{Z}$. Define linear transformations A, B from V to itself by $Ae_n = e_{n+1}$ and $Be_n = 2^n e_n$ for all $n \in \mathbb{Z}$. Determine all linear transformations S from V to itself such that $SA = AS$ and $SB = BS$.

Problem 5. Suppose that n and k are positive integers satisfying $k^n \equiv 1 \pmod{n}$. Prove that

$$\sum_{j=0}^{n-1} k^j \equiv 0 \pmod{n}.$$

Problem 6. Suppose V_1, V_2, W_1, W_2 are four linear subspaces of $\mathbb{R}^n$ that satisfy $V_1 \subseteq V_2$ and $W_1 \subseteq W_2$. Prove that there exists a basis $e_1, \ldots, e_n$ for $\mathbb{R}^n$ such that all four linear subspaces V_1, V_2, W_1, W_2 are spanned by subsets of the basis.

Problem 7. Let m and n be positive integers, with $m \leq n$. Two friends, named A and B, play a cooperative game together in which they take turns saying numbers. Player A begins the game with $a = 0$. After that, player B always takes the last number a said by A, and then reduces $a + m \pmod{n}$ to get a new number b between 0 and $n - 1$. Similarly, player A always takes the last number b said by B, and then reduces $b + n \pmod{m}$ to get a new number a between 0 and $m - 1$. They win the game if player A gets back to 0 at some point. For which pairs of integers $m \leq n$ will they win the game? (For example, if $m = 2$ and $n = 5$, then the two players say $0, 2, 1, 3, 0$ and win.)

Problem 8. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a real-valued function satisfying:

- $f(nx) \geq f(x) \geq 0$ for all real x and positive integer n;
- $f(x + 1) = f(x)$ for all real x;
- $\lim_{x \to 0} f(x) = 0$, but $f(0) = 1$.

At which points is f continuous?

Problem 9. You are trying to climb a very slippery hill. Your height (above the base of the hill) is always a positive integer H. Each second, you try to climb up from height H to height $H + 1$, but you might slip and slide down part of the hill in the process - your new height is equally likely to be any integer from 1 to $H + 1$ (inclusive). Prove that for any posittive integer n, if you start at $H = n$, then the expected number of seconds to reach $H = n + 1$ is less than $e \cdot n!$.

Problem 10. Find all real-valued rational functions $R(t) = P(t)/Q(t)$ such that

- $R'(t) + R(t)^2$ is a polynomial;
- $\lim_{t \to \infty} R(t) = 0$.

Contributors: Ben Baily, Hyman Bass, Stephen DeBacker, Robert Griess, Aaron Pixton, Jennifer Wilson.