Eigenvalues and eigenvectors of combinations of spin operators

Quantum mechanics has a very different mathematical formulation from classical mechanics. The former is based on operators (linear maps), and key related concepts, namely eigenvalues and associated eigenvetors. On the other hand, classical mechanics is formulated in terms of differential geometry. However, in some sense the two mechanics must be related, as both are very successful physical theories (albeit for different processes). The goal of this project is to investigate aspecrts of this relationship, in the context of spin.

For each positive integer N, let $\mathcal{H}_N$ denote the $(N+1)$-dimensional vector space of polynomials in one (complex) variable of degree at most N. To each of the the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad i = \sqrt{-1} \tag{1}$$

there corresponds a linear operator (linear transformation) $S_j : \mathcal{H}_N \to \mathcal{H}_N$. These operators arise in the theory of spin (or of angular momentum) in quantum mechanics, where the number $\frac{N}{2}$ is the spin and the S_j are called spin operators. In geometry, they arise in relation to infinitesimal rotations of the Riemann sphere $\mathbb{S}^2$, which is the complex plane with an added point at infinity. They satisfy the identity

$$S_1S_2 - S_2S_1 = [S_1, S_2] = 2iS_3,$$

and cyclic permutations of the indices. (The S_j depend on N and these relations hold for every N; $S_1S_2 = S_1 \circ S_2$ for example denotes the composition of S_1 and S_2, etc.) The same identities are satisfied by the Pauli matrices. There is a natural inner product on $\mathcal{H}_N$ with respect to which the S_j are Hermitian.

In this project we will study the spectrum and eigenvectors of non-linear combinations of the S_j. For example, the operator

$$P = S_3^2 + c\,(S_1S_2 + S_2S_1) : \mathcal{H}_N \to \mathcal{H}_N$$

with $c \in \mathbb{R}$ is already of interest. In all cases that we will study there exists an orthonormal eigenbasis $\{\psi_j \;:\; j = 0, 1, \dots N\}$ of $\mathcal{H}_N$:

$$\forall j = 0, \dots N \qquad P\psi_j = E_j\psi_j, \quad E_j \in \mathbb{R}.$$

Moreover, for each P there is an associated function

$$H_P : \mathbb{S}^2 \to \mathbb{R},$$

the so-called *symbol* of P. H_P is obtained by replacing each S_i by x_i, the i-th Cartesian coordinate in which $\mathbb{S}^2$ lives. (In the example above, $H_P = x_3^2 + 2cx_1x_2$.) This function H_P is the "classical observable" correspoinding to the "quantum mechanical observable" P.

Roughly speaking, we are interested in studying where the functions $h_j(z) = |\psi_j(z)|^2$ attain their maxima on the Riemann sphere. It is more or less known that the h_j should peak on or near a level curve of H_P as $N \to \infty$. It is in the $N \to \infty$ limit that one expects to see a relationship between the quantum and classical objets. But it is unclear what in fact happens for finite N. This is a difficult problem to study theoretically, and it would be very interesting to study it numerically. There are other related questions for which not much is known, for example, the distribution of the zeros of the ψ_j.

Prerequisites:

- A solid background in linear algebra: Abstract vector spaces, linear transformations, eigenvalues, eigenvectors, bases, matrices of transformations. Some familiarity with complex numbers.
- Differential calculus of multivariable functions.
- Some experience using MATLAB, Mathematica, or another language which can be applied to the problem described above. We will need to compute eigenvalues and eigenvectors of large matrices, transform the eigenvectors into functions $\psi_j(z)$, and plot the h_j.
- No background in mechanics is required.