Strength and geometry of polynomials

For a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, we often want to know whether f is irreducible. One reason is that in this case the set of zeros $\{x \in \mathbb{C}^n : f(x) = 0\}$ is something called an irreducible algebraic set and is better behaved geometrically. A useful *algebraic* measure of complexity is the *strength* of f. If $f = g \cdot h$ then f is reducible and has strength 1. More generally, if $f = g_1 \cdot h_1 + \ldots + g_r \cdot h_r$ then f has strength $\leq r$. Here is a table with some examples of polynomials with complex coefficients and their strength (Can you prove these values are correct?)

Polynomial	Strength
$x^2 + y^2$	1
$xy + zw$	2
$x^{101} + y^{101}$	1
x	∞

Just as the set of zeros of an irreducible polynomial is "nicer", it turns out that the set of zeros $\{x \in \mathbb{C}^n : f(x) = 0\}$ is less singular for f of large strength. In other words, algebraic complexity implies better behavior geometrically.

The goal of this project is to gain a better understanding of the quantitative aspects of this relationship between strength and singularities. One way to do this is by searching for examples to test whether the known bounds are tight. Another option is sampling polynomials randomly in cases where quantitative bounds aren't yet known in order to make a conjecture about the correct bounds. There are also closely related questions for polynomials with coefficients in finite fields. We will choose a project depending on students' interests and background.