LOG(M) PROJECT: STATISTICS OF THE CHARACTER TABLE OF S_n

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1. Overview

A *partition* $\lambda = (\lambda_1, \dots, \lambda_k)$ of a natural number n is a decreasing sequence $\lambda_1 \geq \cdots \geq \lambda_k$ of natural numbers that sums to n. For example, there are seven partitions of 5:

$$(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), \text{ and } (1,1,1,1,1).$$

The conjugacy classes of S_n are indexed, in a natural way, by the partitions of n. Indeed, the conjugacy class of a permutation $\sigma \in S_n$ is determined by its cycle type, and the lengths of the cycles in the cycle decomposition of σ give a partition of n. A (complex) *representation* of a finite group G is a homomorphism from G into $\mathrm{GL}(V)$ for some finite dimensional complex vector space V. There is also a natural and beautiful bijective correspondence between partitions of n and irreducible representations of S_n. Thus, for every natural number n, one can organize the data of all values of irreducible characters on conjugacy classes of S_n in a square table, called the *character table*, with rows and columns indexed by the partitions of n. Character tables of finite groups encode key information, and show up in countless places in math, physics, and chemistry. Here is the character table of S_5:

	(5)	$(4,1)$	$(3,2)$	$(3,1,1)$	$(2,2,1)$	$(2,1,1,1)$	$(1,1,1,1,1)$
(5)	1	1	1	1	1	1	1
$(4,1)$	4	-2	0	1	1	0	-1
$(3,2)$	5	-1	1	-1	-1	1	0
$(3,1,1)$	6	0	1	0	1	0	1
$(2,2,1)$	5	1	1	-1	1	-1	0
$(2,1,1,1)$	4	2	0	1	-1	0	-1
$(1,1,1,1,1)$	1	-1	1	1	-1	-1	1

Note that the entries of this table are all integers. It is a general fact that all entries of the character table of S_n are integers for every n. Thus, it is natural to ask about the statistics of these arrays of integers as n tends to infinity: What proportion of entries are even vs odd? How are entries distributed in congruence classes modulo larger integers? What proportion of entries are zero? What proportion of entries are positive vs negative? How large can entries get? How are the sizes of entries distributed?

In 2017, Miller investigated the first four questions, computing the character tables of S_n for all $n \leq 38$. He observed that, as n increased, the density of even entries seemed to increase, and thus he conjectured that the proportion of even entries in the character table of S_n tends to 1 as n tends to infinity. This conjecture is now a theorem, which holds, more generally, when "even" is replaced by "divisible by m" for any fixed integer m. Based on Miller's data, the proportion of positive and negative entries seem to be roughly equal, but it is unclear whether the proportion of zeros is tending to zero or some positive constant (though very recent Monte Carlo simulations by Miller and Scheinerman suggest that the proportion of zeros in the character table of S_n may be decreasing like $\frac{C}{\log n}$ for some $C > 0$). There is still a lot that we do not understand, even computationally, about the statistics of character tables of S_n.

Character tables of symmetric groups can be computed using the combinatorics of partitions. In this project, students will learn basic facts about the representation theory of symmetric groups and algorithms for computing character values. They will implement these algorithms efficiently, use them to explore the statistical properties of character tables of symmetric groups, and hopefully make new conjectures.

2. Prerequisites

(1) Programming experience (ideally, familiarity with C/C++)
(2) Math 412/493 or equivalent