Probability Distribution of Nonlinear Spin Systems

Hao Pan, Haocheng Zhang, Jialei Wang, Yunxi Yan, Faculty Advisor: Alejandro Uribe-Ahumada, Cyril Cordor

Laboratory of Geometry at Michigan

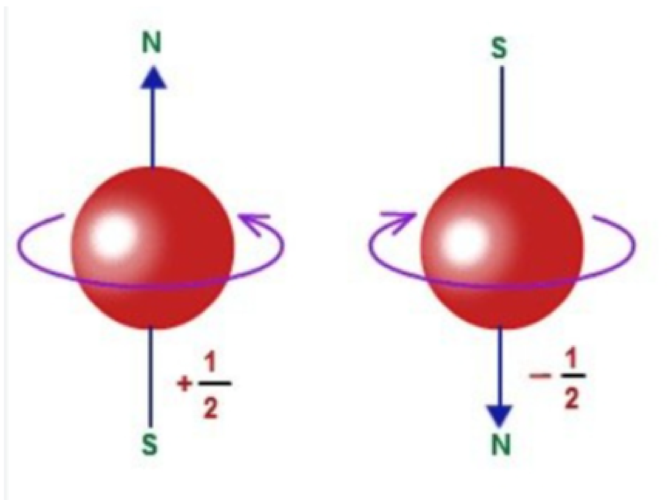
LOG(M)

Introduction

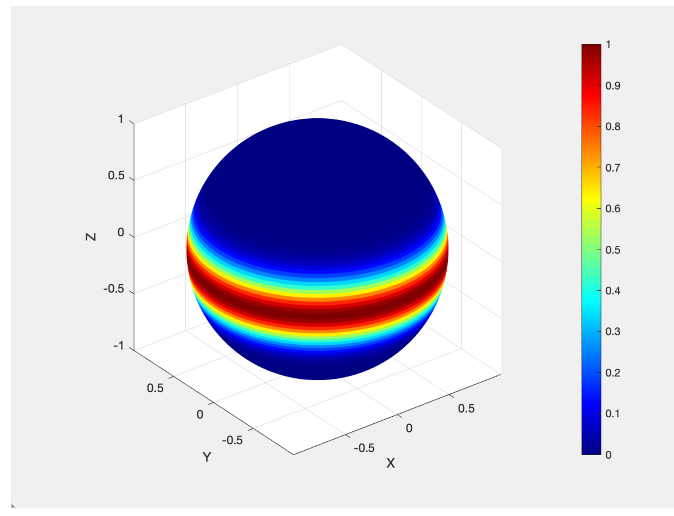
Quantum mechanics has a very different mathematical formulation from classical mechanics, but a very closed physical picture in certain subject. In our project, we will explore the underlying connection between them by examine a "quantum top", which has another fancy name: "spin".

Mysteries of Spin

- magical "rotation" in the quantum world
- observable
- discrete
- random with certain probability distribution

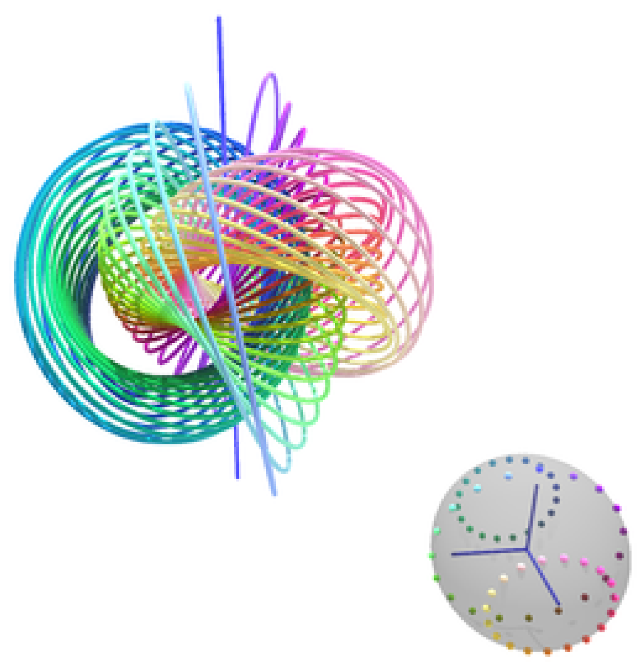


Spin of electrons



Probability distribution on a sphere

Rotaitons in Two Worlds



Complex: $\mathbb{S}^3 \subset \mathbb{C}^2$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{S}^3 \xrightarrow{g \in SU(2)} \mathbb{S}^3 \ni g \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\downarrow \pi \qquad\qquad\qquad \downarrow \pi$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{S}^2 \xrightarrow[R \in SO(3)]{} \mathbb{S}^2 \ni R \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Real: $\mathbb{S}^2 \subset \mathbb{R}^3$

Relation between SU(2) and SO(3)

Research Goal

We will depict and visualize the probability of direction of spins in different nonlinear systems and try to find certain patterns.

Basic Concepts

Definition 1 (The Vector Space V_N): The space of all polynomials $\psi(z_1, z_2)$ of the form $\psi(z_1, z_2) = a_0 z_1^N + a_1 z_1^{N-1} z_2 + \cdots + a_N z_2^N$, where $a_0, a_1, \ldots, a_N$ are complex numbers. This space forms the foundation for exploring quantum spin in mathematical terms.

Definition 2 (The Spin Operators S_j): Operators defined for acting on V_N, given by the expressions:

- $S_1 \psi = \frac{1}{2}\left(z_2 \frac{\partial \psi}{\partial z_1} + z_1 \frac{\partial \psi}{\partial z_2}\right)$,
- $S_2 \psi = \frac{i}{2}\left(-z_2 \frac{\partial \psi}{\partial z_1} + z_1 \frac{\partial \psi}{\partial z_2}\right)$,
- $S_3 \psi = \frac{1}{2}\left(z_1 \frac{\partial \psi}{\partial z_1} - z_2 \frac{\partial \psi}{\partial z_2}\right)$.

These operators are crucial for understanding the dynamics of quantum spin systems.

Methods and Results

We used Matlab to generate functioning codes to explore potential patterns and match each scenarios with corresponding eigenvalues.

function for plotting

- plot the probability distributions on a sphere

Theorem 1

The Husimi function of $|\psi|^2$ composed with Hopf fibration, is given by:

$$h: S^2 \to R, \quad h(\zeta, x_3) = \left(\frac{(1+x_3)^N}{2^N}\right) \left|\psi\left(\frac{\zeta}{1+x_3}, 1\right)\right|^2$$

where ζ is a complex variable and x_3 is a real variable

Cases for S_3 system

Theorem 2

If E_j is an eigenvalue of S_3, then the maximum of the Husimi function concentrates at level curves $\{x_3 = \frac{2E_j}{N}\}$.

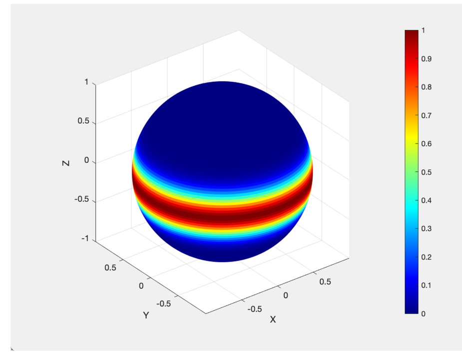


Figure 1: *S_3, N = 30, Zero eigenvalue*

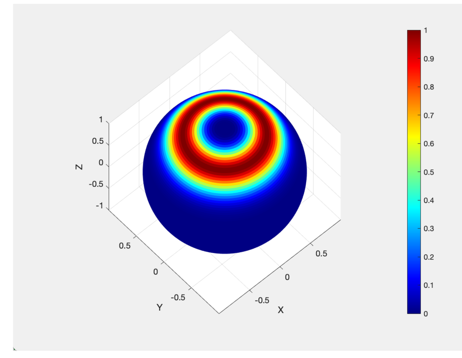


Figure 1: *S_3, N = 30, middle eigenvalue*

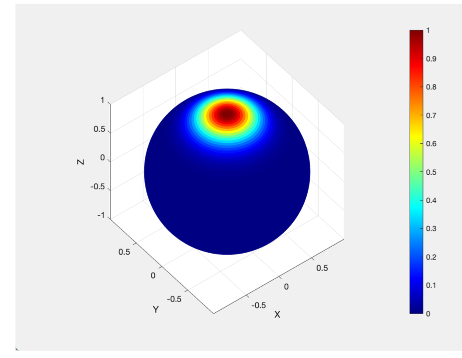


Figure 1: *S_3, N = 30, Smallest eigenvalue*

Hamiltonian System: $H = S_1^2 - S_3^2$

Conjecture: The Husimi function of ψ_j- eigenfunction with eigenvalue E_j accumulates for large N on the level set $\{x_1^2 - x_3^2 = \frac{4}{N^2} E_j\}$

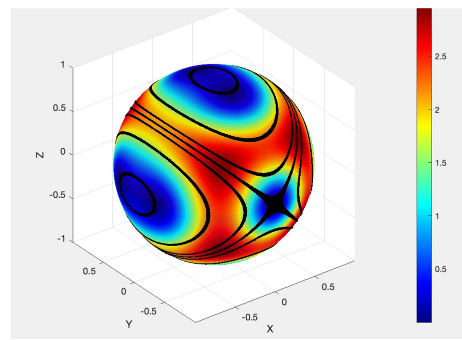


Figure 2: *Hamiltonian System, N = 30, zero eigenvalue*

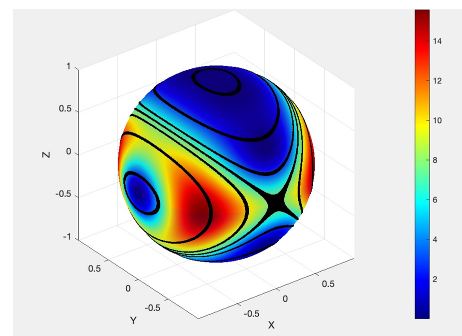


Figure 2: *Hamiltonian System, N = 30, middle eigenvalue*

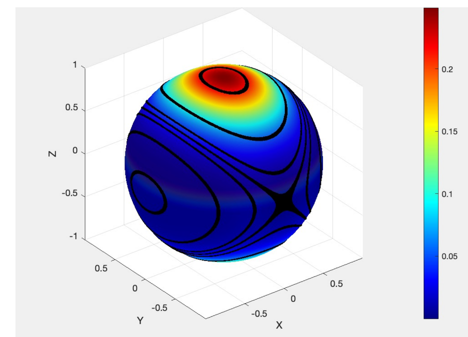


Figure 2: *Hamiltonian System, N = 30, Smallest eigenvalue*

Conclusion: Our conjecture is **PROBLEMATIC**!

Open Question: Can you think of what is the relation between the sphere graph and the level curve?

References

[1] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory*, 3rd Edition, Butterworth-Heinemann, 1977.

[2] Vassanji, PR. Application of matrix mechanics to the asymmetric rotor in the high-spin limit. *Physical Review C*, 19(1): 136, 1979.

[3] Illinois Geometry Lab. IGL Poster Template. *University of Illinois at Urbana-Champaign Department of Mathematics*, 2017.