Statistics of the Character Table of $\mathrm{S_n}$

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1. Introduction

Goal

Compute the character table of S_n , and study its various statistics such as the density of zeros, the congruence property of character values, and the equidistribution phenomenon modulo prime numbers in some specific column.

- Characters form an orthonormal basis for the class functions of a group, the character tables help us classify and understand the irreducible representations of the group.
- They capture the different behaviors of different conjugacy classes of S_n, just like a "periodic table" for symmetric groups.

2. Background

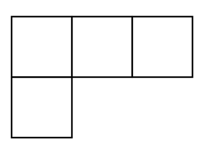
- All the entries in the character tables of S_n are integers.
- In 2017, Miller [1] calculated the entries of all the character tables of S_n for all $n \leq 38$.
- Based on his calculations, various observations were made, including that the density of even entries seemed to tend to 1. This was later proved by Prof. Sarah Peluse [2].
- Miller then conjectured that, even more generally, for any fixed prime power p^k, almost every entry of the character table of S_n is a multiple of p^k as n goes to infinity. This was again proved by Prof. Sarah Peluse in [3].
- The rows and columns of the character table of S_n can be indexed by partitions of n. For more details, please see [4].

Definition. *[4] A **partition** of a positive integer n is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$ and $n = \lambda_1 + \lambda_2 + \cdots + \lambda_l$. Write $\lambda \vdash n$ to denote that λ is a partition of n.*

- The irreducible representations of S_n and conjugacy classes of S_n have one-to-one correspondence with the partitions of n, which can be described by the Young diagrams.

Definition. *[4]*

- *A **Young diagram** is a finite collection of boxes arranged in left-justified rows, with the row sizes weakly decreasing.*
- *The Young diagram associated to the partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ is the one that has l rows, and λ_i boxes on the ith row.*
- *For instance, the Young diagram corresponding to the partition $(3, 1) \vdash 4$ is*



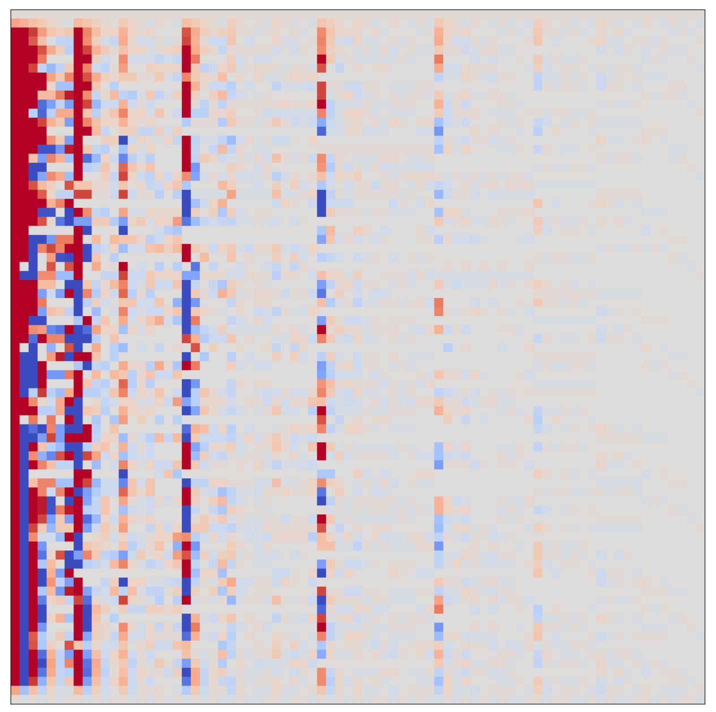
Theorem 1: (Frobenius Formula)

- Given two integer partitions $\lambda, \mu \vdash n$, let χ^λ_μ denote the character value of character corresponding to λ evaluated at conjugacy class μ
- $\Delta(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$
- $P_\mu(x) = P_{\mu_1}(x) P_{\mu_2}(x) \cdots P_{\mu_n}(x)$, $P_0(x) = 1$, $P_k(x) = x_1^k + x_2^k + \cdots x_n^k$ for $k \geq 1$
- $\chi^\lambda_\mu =$ coeff. of $x_1^{n+\lambda_1-1} x_2^{n+\lambda_2-2} \cdots x_n^{\lambda_l}$ in $\Delta(x) P_\mu(x)$

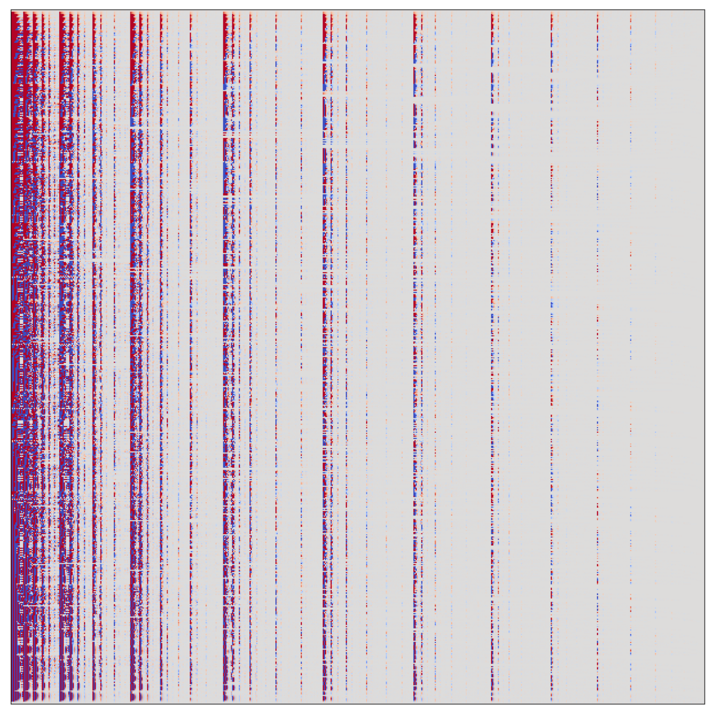
Theorem 2: (Murnaghan-Nakayama Rule)

- $\chi^\lambda_\mu = \sum_{\xi \in \mathrm{bs}(\lambda, \mu_1)} (-1)^{ht(\xi)} \chi^{\lambda \backslash \xi}_{\mu \backslash \mu_1}$
- Base case: $\chi^{()}_{()} = 1$
- The summation ranges over all the border strip ξ that have μ_1 boxes from λ whose removal leaves a valid Young diagram
- $\mu \backslash \mu_1$ = the partition obtained by removing the first element μ_1 from μ.
- $ht(\xi)$ = the height of the border strip = the number of rows that it occupies minus 1.

	(1,1,1,1,1,1)	(2,1,1,1,1)	(2,2,1,1)	(2,2,2)	(3,1,1,1)	(3,2,1)	(3,3)	(4,1,1)	(4,2)	(5,1)	(6)
(6)	1	1	1	1	1	1	1	1	1	1	1
(5,1)	5	3	1	-1	2	0	-1	1	-1	0	-1
(4,2)	9	3	1	3	0	0	0	-1	1	-1	0
(4,1,1)	10	2	-2	-2	1	-1	1	0	0	0	1
(3,3)	5	1	1	-3	-1	1	2	-1	-1	0	0
(3,2,1)	16	0	0	0	-2	0	-2	0	0	1	0
(3,1,1,1)	10	-2	-2	2	1	1	1	0	0	0	-1
(2,2,2)	5	-1	1	3	-1	-1	2	1	-1	0	0
(2,2,1,1)	9	-3	1	-3	0	0	0	1	1	-1	0
(2,1,1,1,1)	5	-3	1	1	2	0	-1	-1	-1	0	1
(1,1,1,1,1,1)	1	-1	1	-1	1	-1	1	-1	1	1	-1



S_{12} *(values truncated within* $[-25, 25]$*)*

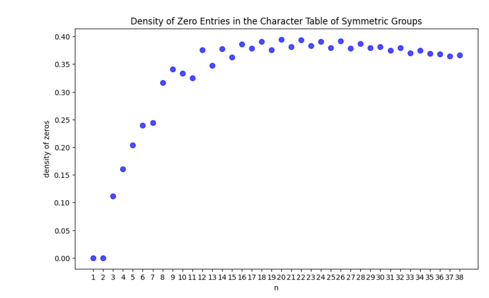


S_{20} *(values truncated within* $[-100, 100]$*)*

3. Results

3.1 Are most entries 0?

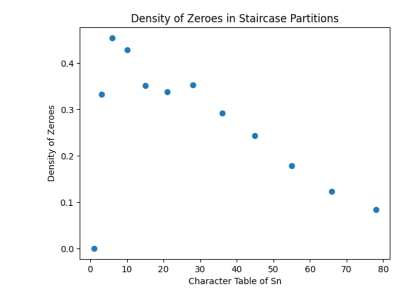
- We can see in Figure 2 that most entries are in a small range ([-2,2] in this case) and a number of them are 0. Is this true for all character tables?
- The density of zeros seems to be decreasing when n is large. However, it remains unknown whether it converges to 0 or some other value.



Zero Density vs Character Table

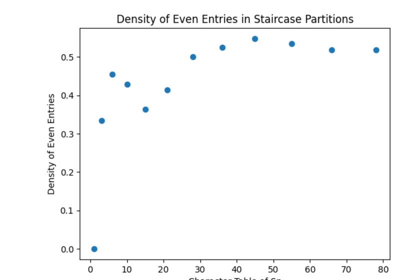
3.2 Staircase Partitions

- Generally, most values in a column tend to be small when there are no repeating parts in that column's partition.
- The staircase partition, i.e. partition of the form $\rho_n = (n, n-1, \ldots, 1)$, have no repeating parts. Are most entries 0?



Zero Density in Staircase Partition Column

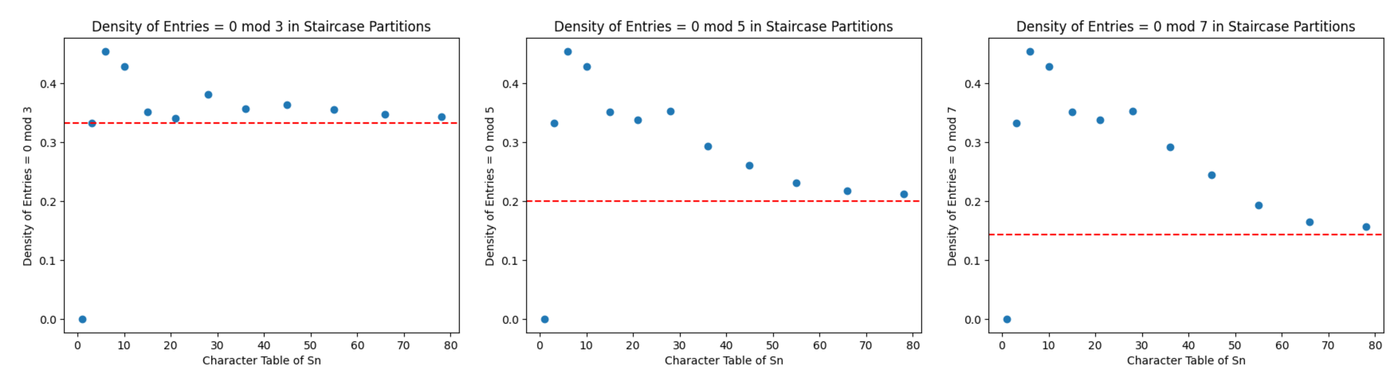
- As mentioned earlier, the density of even entries in the whole character tables approaches 1. However, is this true for specific columns?
- For staircase partitions, turns out that the proportion of even entries tends to $\frac{1}{2}$ according to the statistics.



Even Density in Staircase Partition Column

Conjecture

Let p be a prime number. Then: $\lim_{k \to \infty} \frac{\#\{\mu \vdash \binom{k+1}{2} : \chi^\mu_{\rho_k} \equiv l \pmod p\}}{\#\{\mu : \mu \vdash \binom{k+1}{2}\}} \to \frac{1}{p}, \forall\, 0 \leq l \leq p-1$



Density of entries $\equiv$ *0 mod p where p = 3,5,7*

References

[1] Alexander R. Miller. Note on parity and the irreducible characters of the symmetric group, 2017.

[2] Sarah Peluse. On even entries in the character table of the symmetric group, 2020.

[3] Sarah Peluse and Kannan Soundararajan. Almost all entries in the character table of the symmetric group are multiples of any given prime, 2023.

[4] Yufei Zhao. Young tableaux and the representations of the symmetric group. 01 2008.