LOG(M) PROJECT: BESSEL FUNCTIONS FOR $\mathrm{GL}_n(\mathbb{F}_q)$

ELAD ZELINGHER

Many special functions in analysis have analogs over finite fields, which occur naturally in representation theory. Some of these functions include the beta function, the gamma function, and the Bessel function.

Gelfand and Graev introduced a generalization of the Bessel function, associated to every irreducible representation of $\mathrm{GL}_n(\mathbb{F}_q)$. This Bessel function has great importance in representation theory of $\mathrm{GL}_n(\mathbb{F}_q)$. However, it is quite difficult to compute its values by hand.

The goal of this project is to teach the students how to use SageMath in order to solve problems in abstract math. In this project, the students will compute Bessel functions for irreducible generic representations of $\mathrm{GL}_n(\mathbb{F}_q)$. The computations will be symbolic computations which will be performed using SageMath. The students will learn some basic representation theory of finite groups and some basic knowledge about $\mathrm{GL}_n(\mathbb{F}_q)$ and its representation theory. One of the first tasks the students will perform is compute the Bessel function for $\mathrm{GL}_2(\mathbb{F}_q)$ and see that it is in fact analogous to the classical Bessel function.

Prerequisites.

- Some familiarity with Group Theory (ideally Math 412 Introduction to Modern Algebra).
- Math 217 Linear Algebra or equivalent (ideally Math 420 Advanced Linear Algebra).
- Some coding experience - ideally Python, because SageMath is based on Python, but any other past coding experience would also work.

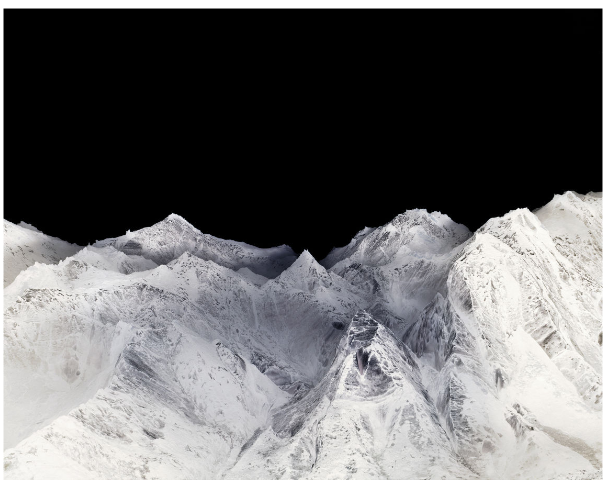
Fractional Brownian Motions

Hai Le

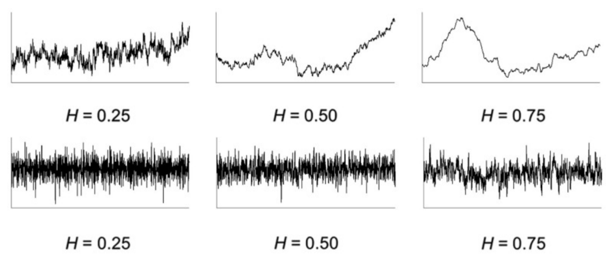
1 The project

The term *Fractional Brownian motions* (fBm's) were first used by Mandelbrot and van Ness in their seminal paper "*Fractional Brownian motions, fractional noises and applications*" to give a generalization to the standard Brownian motions. They realized fBm's sometimes serve as a better model for many random phenomena because of its "interdependence between distant samples". What that means informally is the change in the future will depend on the changes in the past, unlike in the case of standard Brownian motions where the changes are independent of each other. Consequently, fBm's have found applications in modeling random processes appearing in economics, finance, hydrology, wave propagation in random media, etc.

The goal of this project is to introduce its participants to the field of ***stochastic processes***. In particular, we will focus on fractional Brownian motions (and standard fBm's). On the theoretical side, we will learn how to construct fBm's and study their properties under the framework of Gaussian processes. On the numerical side, we will learn how to simulate fBm's and as a prize, use that to draw moutains!



(a) Mountains generated by fBm.[1]



(b) Realizations of fBm's at different Hurst exponents.

2 Prerequisites

- Math 217 Linear Algebra or equivalent
- Some familiarity with probability theory (ideally 425: Introduction to probability or equivalent).
- Some coding experience. We will mainly use Python and Matlab.

[1]The Book of Shaders by Patricio Gonzalez Vivo & Jen Lowe.

Lengths of Quotient Rings

Jenny Kenkel

What is the vector space dimension of the set of all polynomials with variable x of degree d or less? What about the vector space dimension of the set of all polynomials with variables x_1 and x_2, such that x_1 appears with exponent no more than d_1, and x_2 appears with exponent no more than d_2? What if we have n variables, x_1 through x_n, such that x_i appears with exponent no more than d_i for i from 1 to n? We can investigate this question by studying the number of **monomials** (that is, elements of the form $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$) in the quotient ring

$$k[x_1,\ldots,x_n]/(x_1^{n_1},\ldots,x_n^{d_n})$$

where k is some field. Does your answer change if k is a field of positive characteristic, that is, something like $\mathbb{Z}_p$ for some prime number p?

This question has an answer that is relatively straightforward to write down, but adding in just one relation, and instead studying the quotient ring

$$k[x_1,x_2,x_3]/(x_1^{n_1},x_2^{d_2},x_3^{d_3},x_1+x_2+x_3)$$

where k is a field of positive characteristic is a much harder questions! This question has been studied in various works: [RRR91, Han92, HM93, Vra15]. While it is a large problem, to find a closed form equation, it's straightforward to calculate examples via algebra software Macaulay2. We will begin by restricting our attention to a field of characteristic 2, generating many examples using algebra software Macaulay2 and case-by-case investigations. We'll compare these lengths to well-known sequences using the Online Encyclopedia of Integer Sequences [Inc22], and then form conjectures for lengths or bounds that we will then prove. Depending on our success, we may generalize to more variables or other characteristics.
Prerequisites: Math 412 or 493 (or equivalent).

References

[Han92] C. Han. The Hilbert-Kunz function of a diagonal hypersurface. ProQuest LLC, Ann Arbor, MI, 1992. Thesis (Ph.D.)–Brandeis University.

[HM93] C. Han and P. Monsky. Some surprising Hilbert-Kunz functions. Math. Z., 214(1):119–135, 1993.

[Inc22] OEIS Foundation Inc. The on-line encyclopedia of integer sequences, 2022.

[RRR91] L. Reid, L. G. Roberts, and M. Roitman. On complete intersections and their Hilbert functions. Canad. Math. Bull., 34(4):525–535, 1991.

[Vra15] A. Vraciu. On the degrees of relations on $x_1^{d_1},\ldots,x_n^{d_n},(x_1+\cdots+x_n)^{d_{n+1}}$ in positive characteristic. J. Algebra, 423:916–949, 2015.

Oscillatory dynamics of simple circadian models

Ruby Kim

Physiologically-based models using ordinary differential equations (ODEs) can be used to help explain complicated biological phenomena, including circadian (24hr) rhythms that exist within almost every cell in your body. Disruptions to circadian rhythms are linked to various neurological conditions including Parkinson's disease, sleep disorders, and depression. These circadian disruptions occur at both the molecular and behavioral levels.

We will analyze several existing ODE models of circadian rhythms to study the properties required for stable oscillatory behavior. We will use MATLAB to compute numerical solutions and investigate any qualitative changes in these dynamical systems, including the loss of rhythmicity. Prerequisites for this project are Math 217 and some exposure to ordinary differential equations. Coding experience is helpful but not required.

RICCI FLOW AND RICCI SOLITONS

NICHOLAS MCCLEEREY

The Ricci flow was introduced by Hamilton in the '80's as an analouge of the heat equation for Riemannian metrics; later, it was famously used by Perelman to prove the Poincaré conjecture. The flow naturally deforms the metric by "shrinking" parts where the curvature is positive and "pushing out" parts where the curvature is negative. Because of this, the flow collapses many natural shapes in finite time – for instance, a round sphere will shrink down to a point. This collapsing behavior can be problematic if we flow for long periods of time. For example, a dumbbell with a really long handle bar might have the center of the bar pinch before the spherical ends collapse, leaving us with an irregular surface.

To better understand collapsing, one can carefully zoom in near a pinched point as we move along the flow. This gives you a new manifold, which tends to preserve its shape along the flow – this is called a Ricci soliton. Classifying all possible Ricci solitions should then lead to a (more-or-less) complete understanding of how the Ricci flow can collapse.

The proposed project is to investigate explicit examples of Ricci solitions on surfaces and other special manifolds where there is a lot of symmetry. In these cases, the Ricci soliton equation can be reduced to an ODE and often solved explicitly. This will allow us to visualize several interesting geometric effects of the Ricci flow. Exact questions will depend on student background and interest.

Prerequisites:

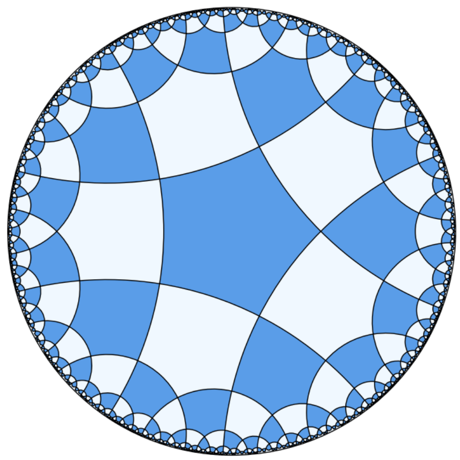
– A class in ODE's
– Some knowledge of Reimannian geometry (e.g. MATH 433) and prefereably complex numbers
– Some programming experience is a plus (preferably Mathematica)

Subgroups of discrete reflection groups

Teddy Weisman

Winter 2023

A *reflection group* is a discrete group of symmetries of some space X generated by *reflections*: involutions of X which fix a unique hyperplane $W \subset X$ (a "wall" or "mirror"), and exchange the two half-spaces on either side of W. Reflection groups abound throughout geometry, and include basic examples like dihedral groups and finite symmetric groups, as well as more complicated groups that generate beautiful tilings of non-Euclidean spaces (for instance hyperbolic space).



Reflection groups are very well-studied partly because of their ubiquity, but also because it's possible to get a very concrete understanding of their structure through the abstract theory of *Coxeter groups*. Possibly the easiest Coxeter groups to understand are the *right-angled Coxeter groups*, which correspond to reflection groups where the "walls" of the reflections meet at right-angles. Despite their relatively simple definition, right-angled Coxeter groups (and their cousins, the right-angled Artin groups) have a rich theory—for instance, they played a key role in the celebrated proof of the virtual Haken and virtual fibering conjectures, deep structural results about the topology of 3-dimensional manifolds.

We will spend a lot of this project learning about the theory of right-angled Coxeter groups, and the way they can be realized as groups of matrices in $\mathrm{GL}(n, \mathbb{Z})$. Depending on time and interest, we may also explore the connection to cube complexes and right-angled Artin groups. Ultimately, the goal would be to try and find interesting *subgroups* of certain right-angled Coxeter groups. Specifically, we will be looking for matrix subgroups which have the *Anosov property*, meaning that the singular value decompositions of elements of the subgroup satisfy a particular exponential growth condition. Anosov subgroups are interesting to researchers in a number of different areas, but actual constructions can be elusive—so the long-term aim of this project would be to provide evidence that right-angled Coxeter groups give a good way to find a wide variety of them.

Prerequisites

- Should be very comfortable with linear algebra (we may have to spend some time learning linear algebra topics beyond what is covered in Math 217)
- Group theory (Math 412 or 493 or equivalent)
- Some programming experience (Python would be ideal, but C/C++, Matlab, Mathematica, etc. all fine)
- Useful, but not required: knowledge of real projective space and basic algebraic topology (fundamental groups, covering spaces, cell complexes).