Log(M): Pipe dreams for quantum Schubert polynomials

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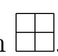
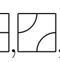
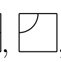
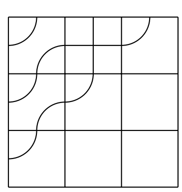
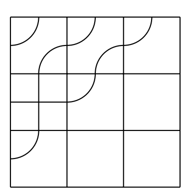
Overview

To every permutation $w\colon \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$, one can associate a *Schubert polynomial* in n variables, denoted $\mathfrak{S}_w(x_1,\ldots,x_n)$. For instance, when $n=3$, there are six such permutations. Writing each permutation as $w(1)w(2)w(3)$, their corresponding Schubert polynomials are given below.

w	$\mathfrak{S}_w$
123	1
213	x_1
132	x_1+x_2
231	x_1x_2
312	x_1^2
321	$x_1^2x_2$

For fixed n, these polynomials form a basis of the *coinvariant ring* of $\mathbb{C}[x_1,\ldots,x_n]$, given by taking the quotient of $\mathbb{C}[x_1,\ldots,x_n]$ by all positive degree polynomials that are symmetric in all n variables. Thus, one can express the product $\mathfrak{S}_u\mathfrak{S}_v$ in the Schubert polynomial basis in the coinvariant ring. In fact, if we write $\mathfrak{S}_u\mathfrak{S}_v = \sum_w c_{u,v}^w \mathfrak{S}_w$ for constants $c_{u,v}^w$, one can show that these constants $c_{u,v}^w$ count how many points lie the intersection of three *Schubert varieties* associated to the permutations u,v,w, and thus $c_{u,v}^w \in \mathbb{Z}_{\geq 0}$. In an effort to better understand these polynomials and the geometric problems they solve, mathematicians have undertaken a *combinatorial* study of these polynomials. One such problem is to find a set of objects $P(w)$ and a weight function $wt\colon P \to \mathbb{C}[x_1,\ldots,x_n]$ sending each object to a suitably chosen monomial such that $\mathfrak{S}_w = \sum_{p\in P(w)} wt(p)$. In fact, for Schubert polynomials, there are many such sets P, but the prototypical set is the set of *pipe dreams* associated to w, denoted $PD(w)$.

These pipe dreams are specified by filling an $n\times n$ grid with 1×1 tiles of the form , , , and  subject to certain rules. Without giving a rigorous definition, the set $PD(132)$ is given by

 and 

In recent years, there have been many generalizations of the Schubert polynomials. One such generalization is given by the *quantum Schubert polynomials*, which have an extra set of "quantum" variables $q_1,\ldots,q_{n-1}$. These polynomials are more intricate than Schubert polynomials, but carry much more information. However, their combinatorics are not understood as well as the usual Schubert polynomials. In particular, there is no known "pipe dream formula" for these quantum

Schubert polynomials. The rough goal of this project is to explore objects that could serve as pipe dreams for quantum Schubert polynomials using computer experimentation.

Goals

- Implement pipe dreams in Python or Sage in a flexible enough way to introduce new tiles and weights for pipe dreams.
- Use this code to formulate and test some conjectures about quantum Schubert polynomials.

Prerequisites

- Math 217 Linear algebra or some equivalent: familiarity with vector spaces and bases.
- (Optional but recommended) Math 412 Modern algebra: some familiarity with the idea of a ring will be helpful, but not strictly necessary.
- Programming experience and a willingness to use Python and/or Sage.