Logarithmic CFTs and the bootstrap

Matthijs Hogervorst

C.N. Yang Institute for Theoretical Physics, SUNY Stony Brook

September 14, 2016

MCTP at Ann Arbor, MI; Brown Bag seminar

arXiv:1605.03959 with M. Paulos and A. Vichi

Lightning review of Conformal Field Theory

- CFTs are crucial to understand the landscape of QFTs.
  In the UV, they encode — in principle — all info about RG flows.
  In the IR, they describe the dynamics of critical points.
- By definition, CFTs are invariant under

$$\left.\begin{array}{l}\text{Poincaré} \\ \text{dilatations } x \mapsto \lambda x \\ \text{special transformations}\end{array}\right| \begin{array}{c} P_\mu,\ M_{\mu\nu} \\ D \\ K_\mu \end{array} \quad = \; SO(d,2)$$

- Good observables are correlators of [renormalized, composite] operators $\mathcal{O}_i$.
  They are characterized by a scaling dimension Δ_i:

$$i[D, \mathcal{O}_i] = \Delta_i \mathcal{O}_i \, .$$

- Correlation functions of the $\mathcal{O}_i$ are simple power laws i.e.

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(y) \rangle = \frac{\delta_{ij}}{|x-y|^{2\Delta_i}} \, .$$

Bootstrapping (1)

- The $\mathcal{O}_i$ satisfy an operator algebra:

$$\mathcal{O}_i \times \mathcal{O}_j = \sum_k c_{ijk}\, \mathcal{O}_k\,.$$

This is really a convergent short-distance expansion (OPE).

- Together with the Δ_i, these c_{ijk} are only local observables:

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \frac{c_{ijk}}{|x_1-x_2|^{\#_1}|x_1-x_3|^{\#_2}|x_2-x_3|^{\#_3}}\,.$$

- Associativity leads to an infinite set of consistency conditions:

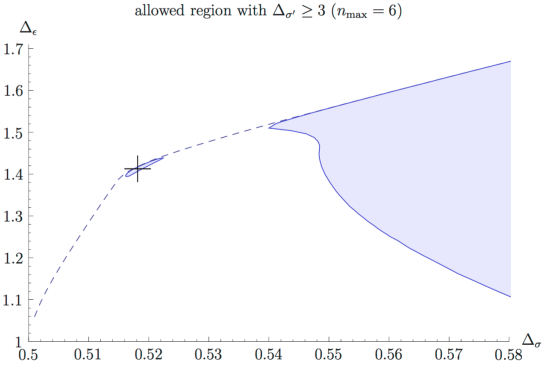
$$\langle \mathcal{O}_i\mathcal{O}_j\mathcal{O}_k\mathcal{O}_l\rangle \sim \sum_n c_{ijn}\, c^n{}_{kl}\ \ldots\ = \sum_n c_{iln}\, c^n{}_{jk}\ \ldots$$

This is the bootstrap principle.

Bootstrapping (2)

- Conclusion: the $c_{ijk} \in \mathbb{R}$ can not be chosen at will. Easy to check if a choice of c_{ijk} satisfies bootstrap equations!
- Can be turned into a method to construct CFTs, and to compute scaling dimensions = critical exponents.

E.g. $3d$ critical Ising [Kos, Poland, Simmons-Duffin 2014]



Logarithmic CFTs

- Some CFTs are more delicate. Their correlators have logs:

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{1}{|x|^{2\Delta}}\left[c_1 + c_2 \ln \mu^2 x^2 + \ldots\right]$$

  that contain a scale (!) μ.
- RG explanation: the matrix of anomalous dimensions cannot be diagonalized at the critical point, but has Jordan blocks.

$$\Gamma = \begin{pmatrix} \gamma & 1 \\ 0 & \gamma \end{pmatrix}.$$

- To get Jordan blocks, fine-tuning is required.
  At a "generic" critical point, no such degeneracies present in spectrum.

Why do we care?

- Unitarity is violated. No way to get logCFTs starting from a healthy Lagrangian with real couplings.
- In the world of statistical physics, they are common. Often constructed through analytic continuation of good CFTs.
- Example: percolation = $Q \to 1$ state Potts model. For generic $Q \in \mathbb{N}$, global symmetry is group S_Q. Different irreps such as

$$\phi_a(\sigma) = \delta_{a,\sigma} - 1/Q \quad \text{and} \quad \tilde{\phi}(\sigma) = 1$$

collide when taking limits $Q \to$ integer.
More complicated "watermelon" operators collide when $Q \to 1$.

[Jacobsen, Saleur, Vasseur]

Why do we care? (2)

- Many more examples:
  self-avoiding walks/polymers $= O(n \to 0)$ model,
  quenched disorder ($n \to 0$ replicas of theory).
- Proving that such limits are logarithmic only uses rep theory.
  Valid for all $d < d_c$, depending on universality class.

  [Cardy]
- Attempt to bootstrap $3d$ $O(n \to 0)$ using conventional techniques.

  [Hikami, Shimada]

  This is not completely kosher: need unitarity, but $O(n)$ model is non-unitary for $n < 1$. Hard to estimate errors.

What is known?

LogCFTs have been intensively studied in $d = 2$ (or 1+1) dimensions.
In this setting, conformal symmetry is much more constraining.

2d toolkit contains:

- Representation theory.
  Logarithmic minimal models $LM(p, q)$ have intricate structure:
  ∞ many Virasoro reps but rational under $\mathcal{W}$-symmetry.
- Spin chains, loop models, integrability ...
  Gives handle on spectrum, fusion rules etc. Some exact or high-precision numerical predictions.

Still a (very) limited understanding of "bulk" physics = chiral+anti-chiral correlators.

Where next?

- In higher d, none of these methods apply.
  But $3d$ percolation, polymers etc. *are* prime examples of CFTs.
- Would be great to attack problem using bootstrap paradigm.
- Two problems to tackle:
  (1) find counterpart of bootstrap equations, and
  (2) decompose them.
  Counterpart of the c_{ijk} coefficients?

Some previous work about logCFT correlators, mostly $SL(2,\mathbb{R})$ constraints in $2d$.

[Flohr "Bits and pieces" review, see also Ghezelbash, Karimipour]

Logarithmic multiplets

- Key role played by "logarithmic multiplets" $\{\mathcal{O}_a\}$ of a given rank $r > 1$. For $r = 2$ we would have

$$D \cdot \begin{pmatrix} \mathcal{O}_1 \\ \mathcal{O}_2 \end{pmatrix} = \begin{pmatrix} \Delta & 1 \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} \mathcal{O}_1 \\ \mathcal{O}_2 \end{pmatrix}.$$

  Hence $\mathcal{O}_2$ transforms like a normal conformal operator under dilatations

$$\mathcal{O}_2(\lambda x) = \lambda^{-\Delta} \mathcal{O}_2(x)$$

  whereas $\mathcal{O}_1$ transforms as

$$\mathcal{O}_1(\lambda x) = \lambda^{-\Delta} \left[\mathcal{O}_1(x) - \ln(\lambda)\, \mathcal{O}_2(x)\right].$$

- Same idea for higher operators of higher rank or with non-zero spin.

Two-point functions

- Let's find most general set of two-point functions

$$\langle \varphi_a(x)\varphi_b(0)\rangle = \frac{B_{ab}(x^2)}{x^{2\Delta}}.$$

- B_{ab} constrained up to r constants, e.g. for $r = 2$

$$B_{ab} = \begin{pmatrix} k_1 - k_2 \ln x^2 & k_2 \\ k_2 & 0 \end{pmatrix}.$$

- Then **either** $k_2 = 0$ and φ_2 decouples **or** we can redefine

$$B_{ab} = k_\varphi \begin{pmatrix} -\ln x^2 & 1 \\ 1 & 0 \end{pmatrix}.$$

Regardless of sign of k_φ, unitarity is violated.

Scale dependence

- All logs must be dimensionless — $\ln x^2$ does not make sense.
- In any actual computation this would be obvious.
  Would have $\ln(\mu^2 x^2)$ in MS or $\ln(x^2/a^2)$ on the lattice.
- Changing μ numerically changes correlators (Callan-Symanzik eqn).
  Yet there is a large ($r-1$ parameter) ambiguity in defining log multiplets.
  Can "undo" change in μ this way by rotation in Hilbert space.
- Will set $\mu = 1$ from now on.

Three-point functions (1)

- More challenging:

$$\langle \mathcal{O}^i_a(x_1)\mathcal{O}^j_b(x_2)\mathcal{O}^k_c(x_3)\rangle = K_{abc}(x_{ij}) \times \frac{1}{|x_{12}|^{\#}|x_{13}|^{\#}|x_{23}|^{\#}}.$$

  In normal CFT, K_{abc} would be a c-number, c_{ijk}.

- To give a taste of the problem: consider $2d$ triplet model [Gaberdiel, Kausch]

$$\langle \omega(x_1)\omega(x_2)\omega(x_3)\rangle = 48(\ln 2)^2 + 8\ln 2\,(\circ\!-\!\circ) + 2\,(\circ\!-\!\circ\!-\!\circ) - (\circ\!=\!\circ)$$

$$\begin{aligned} (\circ\!-\!\circ) &= \sum_{ij} \ln|x_{ij}|^2\,, \\ (\circ\!-\!\circ\!-\!\circ) &= \sum_{ijk} \ln|x_{ij}|^2 \ln|x_{jk}|^2\,, \\ (\circ\!=\!\circ) &= \sum_{ij} \left(\ln|x_{ij}|^2\right)^2\,, \end{aligned}$$

Three-point functions (2)

- Ward identities become messy. Useful to introduce new coordinates

$$\tau_1 = \ln \frac{|x_{23}|}{|x_{12}||x_{13}|}, \qquad \tau_2, \tau_3 = \text{cyclic perms of } \tau_1.$$

  The τ_i have various beautiful properties.

- Then

$$\frac{\partial}{\partial \tau_1} K_{abc} = K_{(a+1)bc}, \quad \ldots, \quad \frac{\partial}{\partial \tau_3} K_{abc} = K_{ab(c+1)}.$$

  Solution is polynomial in the τ_i; finite number of undetermined constants.

Three-point functions (3)

- Example: take rank-two field $\{\varphi_1, \varphi_2\}$:

$$\langle \varphi_a(x_1)\varphi_b(x_2)\varphi_c(x_3)\rangle = K_{abc}(\tau_i) \times \frac{1}{|x_{12}|^\Delta |x_{13}|^\Delta |x_{23}|^\Delta}.$$

- Conformal + Bose (permutation) symmetry at work:

$$\begin{aligned}
K_{222} &= c^{(4)} \\
K_{122} &= c^{(3)} + c^{(4)}\,\tau_1 \\
K_{112} &= c^{(2)} + c^{(3)}\,(\tau_1 + \tau_2) + c^{(4)}\,\tau_1\tau_2 \\
K_{111} &= c^{(1)} + c^{(2)} \sum_i \tau_i + c^{(3)} \sum_{i<j} \tau_i\tau_j + c^{(4)}\,\tau_1\tau_2\tau_3.
\end{aligned}$$

- Simple bookkeeping due to τ variables. No need to keep track of logs.

OPE coefficients

- Recall: in normal CFT, the c_{ijk} show up in the OPE

$$\mathcal{O}_i(x)\mathcal{O}_j(0) \sim \sum_k \frac{1}{|x|^{\#}} c_{ijk}\, \mathcal{O}_k(0) + \text{derivatives of } \mathcal{O}_k.$$

- The coefficients we found above play the same role in logCFT.
  If needed, can write down formulas that look like

$$\mathcal{O}^i_a(x)\mathcal{O}^j_b(0) \sim \sum_k \frac{1}{|x|^{\#}} \left[c^{(1)}_{ijk} + c^{(2)}_{ijk} \ln x^2 + \ldots \right] \mathcal{O}_k(0) + \text{derivatives of } \mathcal{O}_k.$$

  The $c^{(n)}_{ijk}$ are in 1-1 correspondence with three-point functions.
- Never needed in practice.

Four-point functions

- In normal CFT, a 4-pt function depends only on two cross ratios u, v:

$$\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle = F(u,v) \times \text{scale factor}$$

  Bootstrap is analysis of the crossing symmetry relations

$$F(u,v) = F(v,u) = F(u/v, 1/v).$$

- In logCFTs, much more complicated. State-of-the-art results unfit for bootstrap, e.g. $2d$ chiral example: [Flohr, Krohn 2005]

$$\begin{aligned}
\langle 1111\rangle \;=\;& F_{1111} + \mathcal{P}_{(1234)}\Big\{\big[(-\ell_{12}-\ell_{34}+\ell_{23}+\ell_{14})C_1 + (\ell_{13}+\ell_{24}-\ell_{12}-\ell_{34})C_2 \\
&\qquad -\ell_{14}+\ell_{34}-\ell_{13})\big]F_{0111}\Big\} \\
+\;& \mathcal{P}_{(12)(34)}\Big\{\big[(\ell_{13}^2+\ell_{24}^2-\ell_{14}^2-\ell_{23}^2+2(-\ell_{34}\ell_{24}-\ell_{12}\ell_{24}+\ell_{34}\ell_{14}+\ell_{13}\ell_{24} \\
&\quad -\ell_{13}\ell_{34}+\ell_{23}\ell_{34}+\ell_{12}\ell_{23}-\ell_{12}\ell_{13}-\ell_{23}\ell_{14}+\ell_{12}\ell_{14}))C_3 \\
&+(-(\ell_{23}+\ell_{14})^2+\ell_{23}\ell_{34}+\ell_{12}\ell_{14}-\ell_{13}\ell_{34}+\ell_{34}\ell_{14}+\ell_{13}\ell_{14} \\
&\quad -\ell_{34}\ell_{24}-\ell_{12}\ell_{13}-\ell_{12}\ell_{24}+\ell_{23}\ell_{24}+\ell_{23}\ell_{13}+\ell_{12}\ell_{23}+\ell_{24}\ell_{14}))C_4 \\
&-\ell_{34}^2-\ell_{23}^2-\ell_{14}^2+2\ell_{23}\ell_{34}+2\ell_{34}\ell_{14}-2\ell_{12}\ell_{34}-\ell_{23}\ell_{14}+\ell_{23}\ell_{24} \\
&\quad -\ell_{12}\ell_{13}+\ell_{12}\ell_{14}+\ell_{12}\ell_{23}-\ell_{12}\ell_{24}+\ell_{13}\ell_{14}+\ell_{13}\ell_{24})\big]F_{1100}\Big\} \\
+\;& \big[2(\ell_{12}\ell_{24}\ell_{14}-\ell_{23}\ell_{13}\ell_{14}+\ell_{23}\ell_{34}\ell_{24}-\ell_{24}\ell_{13}\ell_{34}-\ell_{23}\ell_{34}\ell_{14} \\
&\quad -\ell_{12}\ell_{23}\ell_{34}-\ell_{12}\ell_{34}\ell_{24}-\ell_{23}\ell_{13}\ell_{24}+\ell_{12}\ell_{23}\ell_{13}+\ell_{13}\ell_{34}\ell_{14} \\
&\quad -\ell_{13}\ell_{14}\ell_{24}-\ell_{23}\ell_{24}\ell_{14}-\ell_{12}\ell_{13}\ell_{24}-\ell_{12}\ell_{23}\ell_{14}-\ell_{12}\ell_{13}\ell_{34}-\ell_{12}\ell_{34}\ell_{14}) \\
&+2(\ell_{13}^2\ell_{24}+\ell_{12}^2\ell_{34}+\ell_{14}^2\ell_{23}+\ell_{23}^2\ell_{14}+\ell_{34}^2\ell_{12}+\ell_{24}^2\ell_{13})\big]F_0\,.
\end{aligned}$$

Four-point functions (2)

- Ansatz for logarithmic case (WLOG):

$$\langle \varphi_a(x_1)\varphi_b(x_2)\varphi_c(x_3)\varphi_d(x_4)\rangle = F_{abcd}(u, v, ?) \times \text{scale factor.}$$

- Play same game as with 3-pt functions. Exchange $\ln |x_{ij}|^2$ for

$$\zeta_1 = \frac{1}{3} \ln \frac{|x_{23}||x_{24}||x_{34}|}{|x_{12}|^2|x_{13}|^2|x_{14}|^2}, \quad \zeta_2, \zeta_3, \zeta_4 = \text{cyclic perms.}$$

  The ζ_i generalize the τ_i from before.

- Again the $F_{abcd}(u, v, \zeta_i)$ obey PDEs in ζ_i — polynomial solution.

Four-point functions (3)

Consider again a rank-2 scalar $\{\varphi_1, \varphi_2\}$. Bose symmetry + conformal invariance combined leave us with 5 undetermined functions $F_n(u,v)$:

$$
\begin{aligned}
\langle 2222 \rangle &= F_5(u,v) \\
\langle 1222 \rangle &= F_4(u,v) + \zeta_1 \, F_5(u,v) \\
\langle 1122 \rangle &= F_3(u,v) + (\zeta_1 + \zeta_2) \, F_4(u,v) + \zeta_1 \zeta_2 \, F_5(u,v) \\
\langle 1112 \rangle &= F_2(u,v) + \zeta_3 \, F_3(u,v) + 2 \text{ terms} \\
&\quad + [\ldots] \, F_4(u,v) + \zeta_1\zeta_2\zeta_3 \, F_5(u,v) \\
\langle 1111 \rangle &= F_1(u,v) + \sum_i \zeta_i \, F_2(u,v) + [\ldots] \, F_3(u,v) + 2 \text{ terms} \\
&\quad \sum_{i<j<k} \zeta_i \zeta_j \zeta_k \, F_4(u,v) + \zeta_1\zeta_2\zeta_3\zeta_4 \, F_5(u,v).
\end{aligned}
$$

All of the $F_n(u,v)$ [except F_3] must obey the crossing relations:

$$F_n(u,v) = F_n(v,u) = F_n(u/v, 1/v).$$

Conformal block decomposition

- Final ingredient in bootstrap is existence of a partial wave decomposition.
- This relates four-point functions to the coefficients c_{ijk}:

$$\langle \varphi\varphi\varphi\varphi \rangle \sim F(u,v) \;=\; \sum_{\mathcal{O}} {c_{\varphi\varphi\mathcal{O}}}^2 \, G_{\mathcal{O}}(u,v)$$

where $G_{\mathcal{O}}(u,v)$ is a known function — a "conformal block" — that only depends on quantum numbers Δ, ℓ of $\mathcal{O}$.

- Same applies to logCFTs. For the aficionados: define the more general blocks

$$\widehat{G}_{\Delta,\ell}(u,v) = u^{-(\Delta_1+\ldots+\Delta_4)/6} v^{(-\Delta_1+2\Delta_2+2\Delta_3-\Delta_4)/6} \times\; G_{\Delta,\ell}(u,v;\Delta_1-\Delta_2,\Delta_3-\Delta_4).$$

Then the logarithmic blocks are derivatives of $\widehat{G}$ w.r.t. $\Delta_1,\ldots,\Delta_4$ and Δ.

Examples of conformal block decompositions

- Example: let φ be a normal scalar and $\{\mathcal{O}_1, \mathcal{O}_2\}$ be an exchanged rank-2 operator. The three-point functions are

$$\langle \varphi\varphi\mathcal{O}_1 \rangle \sim c_1 + \tau_3\, c_2, \quad \langle \varphi\varphi\mathcal{O}_2 \rangle \sim c_2.$$

  The contribution of $\mathcal{O}_i$ to the 4-pt function is

$$\langle \varphi\varphi\varphi\varphi \rangle \sim F(u,v) \supset \left[2c_1c_2 + c_2^2 \frac{\partial}{\partial\Delta}\right] \widehat{G}_{\Delta,\ell}(u,v).$$

- Can treat as sum of separate blocks with coefficients c_1c_2 and c_2^2. But not necessarily > 0:
  (1) the c_i may not be real-valued and
  (2) even if they are real, c_1c_2 not sign-definite.
- Generalizes to higher rank and/or logarithmic external operators.

AdS constructions

- LogCFTs can be realized in AdS – idea from early days of holography

  [Ghezelbash, Khorrami, Aghamohammadi; Kogan]
- Key idea: CFT operators couple to bulk fields with higher-order EoM:

$$S_{\text{bulk}}[\phi] \sim \int d^{d+1}x\sqrt{g}\,\phi\,(\Box - m^2)^r\,\phi + \text{other fields} + \text{interactions}.$$

  Logarithmic boundary conditions possible.
- Can be used to check the formalism from this talk: two- and three-pt functions with tunable couplings, conformal block decompositions etc.
- Various examples known, mostly AdS_3/CFT_2, also higher d.

  [see Grumiller et al. review]

Recap and outlook

- LogCFTs in d dimensions can be tamed!
- Simple way to write down bootstrap equations for general logCFT.
- Numerics: non-unitarity not a problem in principle — use determinant method. Remnants of positivity?
- Can now explore landscape of logCFTs and hunt for kinks. How about "random bond" Ising model? [Komargodski, Simmons-Duffin]
- Many qualitative and quantitative questions are wide open. Time to roll up our collective sleeves!