Name: _____

University of Michigan Physics Department Graduate Qualifying Examination

Part II - Modern Physics

Saturday, January 15, 2005 9:00am-1:00pm

This is a closed book exam - but you may use the materials provided at the exam. If you need to make an assumption or estimate, indicate it clearly. Show your work in an organized manner to receive partial credit for it.

You must answer the first 8 obligatory questions and two of the optional four questions. Indicate which of the latter you wish us to grade (e.g., circle the question number). We will only grade the indicated optional questions. Good Luck.

SOME FUNDAMENTAL CONSTANTS IN CONVENIENT UNITS

$$
\begin{aligned}
\text{speed of light} \quad c &= 2.998 \times 10^8 \text{ m/s} \\
\text{electron charge} \quad e &= 1.602 \times 10^{-19} \text{ C} \\
\text{Planck's constant} \quad h &= 6.626 \times 10^{-34} \text{ J} \cdot \text{s} = 4.136 \times 10^{-15} \text{ eV} \cdot \text{c} \\
\hbar = h/2\pi &= 1.055 \times 10^{-34} \text{ J} \cdot \text{s} = 0.658 \times 10^{-15} \text{ eV} \cdot \text{s} \\
\text{Rydberg constant} \quad R_\infty &= 1.097 \times 10^6 \text{ m}^{-1} \\
\text{Coulomb constant} \quad k = (4\pi\epsilon_0)^{-1} &= 8.988 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2 \\
\text{Universal gas constant} \quad R &= 8.3\text{J/K} \cdot \text{mol} \\
\text{Avogadro's number} \quad N_A &= 6 \times 10^{23} \text{ mol}^{-1} \\
\text{Boltzmann's constant} \quad k_B = R/N_A &= 1.38 \times 10^{-23} J/K = 8.617 \times 10^{-5} \text{eV/K} \\
\text{Stefan} - \text{Boltzmann constant} \quad \sigma &= 5.6703 \times 10^{-8} \text{ W/m}^2\text{K}^4 \\
\text{radius of the sun} \quad R_{sun} &= 6.96 \times 10^8 \text{ m} \\
\text{radius of the moon} \quad R_{moon} &= 1.74 \times 10^6 \text{ m} \\
\text{radius of the earth} \quad R_{earth} &= 6.37 \times 10^6 \text{ m} \\
G_N &= 6.67 \times 10^{-11} m^3/kg/s^2 = 6.71 \times 10^{-39} GeV^{-2}
\end{aligned}
$$

SOME USEFUL CONVERSIONS AND COMBINATIONS

$$
\begin{aligned}
\text{fine structure constant} \quad \alpha &= ke^2/\hbar c = 1/137 \\
\text{Bohr magneton} \quad e\hbar/2m_e &= 9.27 \times 10^{-24} \text{J/T} = 5.79 \times 10^{-5} \text{eV/T} \\
hc &= 19.865 \times 10^{-26} \text{ J} \cdot \text{m} = 12.41 \times 10^3 \text{ eV} \cdot \text{Å} = 1241 \text{ MeV} \cdot \text{fm} \\
\hbar c &= 3.165 \times 10^{-26} \text{ J} \cdot \text{m} = 1973 \text{ eV} \cdot \text{Å} = 197.3 \text{ MeV} \cdot \text{fm} \\
ke^2 &= 1.44 \text{ MeV} \cdot \text{fm} \\
& 1\text{Å} = 10^{-10} \text{ m} = 10^5 \text{ fm} \qquad 1\text{eV} = 1.602 \times 10^{-19} \text{ J}
\end{aligned}
$$

SOME USEFUL RELATIONS

$$\frac{C_V}{Nk} = 9\,(T/\theta_D)^3 \int_0^{\theta_D/T} \frac{x^4 e^4 dx}{(e^x - 1)^2} \quad \text{(Debye formula)}$$

$$= 3\left[1 - \frac{1}{20}\left(\frac{\theta_D}{T}\right)^2 + \ldots\right] = \frac{12\pi^4}{5}\left(\frac{T}{\theta_D}\right)^3 (1 + \ldots)$$

$$\frac{U}{N} = \frac{3}{5}E_F\left[1 + \frac{5\pi^2}{12}\left(\frac{kT}{E_F}\right)^2 - \frac{\pi^4}{16}\left(\frac{kT}{E_F}\right)^4 + \ldots\right] \quad \text{(degenerate electron gas)}$$

$$E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} \quad \text{(nonrelativistic Fermi energy)}$$

$$r = r_\circ A^{1/3}, \qquad r_\circ = 1.2 \times 10^{-15} m \quad \text{(approximate average nuclear radius)}$$

$$E_b = C_1 A - C_2 A^{2/3} - C_3 \frac{Z(Z-1)}{A^{1/3}} - C_4 \frac{(N-Z)^2}{A} \quad \text{(semiempirical binding energy of a nucleus)}$$

$$\frac{n}{s} + \frac{n\prime}{s\prime} = \frac{n\prime - n}{R} \text{(refraction of paraxial rays)}$$

MASSES OF SOME ELEMENTARY PARTICLES

	Rest Mass, m_0 (kg)	m_0c^2 (MeV)
Electron	9.109×10^{-31}	0.511
Proton	1.673×10^{-27}	938.3
bNeutron	1.675×10^{-27}	939.6
Atomic mass unit (1 amu)	1.661×10^{-27}	931.5

VISIBLE LIGHT SPECTRUM

300	400	500		600		700(Nanometer)	
← Ultraviolet	Violet	Blue	Green	Yellow	Orange	Red	Infrared →

Part A: Obligatory Problems

1. Two identical spin 1/2 fermions with mass m are placed in an infinite square well potential:

$$V(x) = \begin{cases} 0, & 0 < x < a \\ \infty, & \text{otherwise.} \end{cases} \tag{1}$$

Write the ground-state wave function and the ground-state energy when the two particles are in a singlet spin state and calculate the first-order correction to the ground-state energy due to the particle-particle interaction:

$$V'(x_1, x_2) = -\alpha\delta(x_1 - x_2)$$

where α is a positive constant. Note that $\int_0^\pi \sin^4\theta d\theta = 3\pi/8$.

2. A spin 1/2 particle is initially in a state

$$\chi = \frac{1}{\sqrt{2}}(\chi_+ + \chi_-)$$

where χ_+ and χ_- are the normalized eigenstates of $\hat{S}_z$ with eigenvalues $+\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$ respectively. Consider the three *consecutive* measurements of

(a) the x-component (S_x)

(b) the z-component (S_z)

(c) the x-component (S_x)

of the spin of the particle, find all possible values and the probabilities of getting those values for each measurement.

3. **Fermi oscillator:** A Fermi oscillator is a system with Hamiltonian $H = f^\dagger f$, where f is an operator that satisfies

$$f^2 = 0, \quad ff^\dagger + f^\dagger f = 1.$$

(a) Show that $H^2 = H$.

(b) Find the eigenvalues of H.

(c) If $H|0\rangle = 0$ and $H|1\rangle = |1\rangle$ and $\langle 0|0\rangle = \langle 1|1\rangle = 1$, what are the states $f|0\rangle$ and $f^\dagger|0\rangle$?

4. Consider a quantum gas of ultra-relativistic fermions with g possible polarizations. Compute the energy density. *Reminder: In the corresponding computation for photons an infinite sum appears which is given exactly as*

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

5. Consider a dilute gas made up of molecules which have a permanent electric dipole moment μ. Compute as a function of temperature

(a) the average polarization $P = \frac{N}{V}\langle \mu \cos\theta \rangle$ of the gas (here θ is the angle between molecule and an external $\mathbf{E}$-field).

(b) the dielectric constant of the gas for weak electric fields. (Recall that the dielectric constant is defined through $\epsilon \mathbf{E} = \mathbf{E} + 4\pi \mathbf{P}$.)

6. The basic two-component Hamiltonian of the Helium atom is given by

$$H = \frac{\mathbf{p}_1^2}{2m} + \frac{\mathbf{p}_2^2}{2m} - \frac{Ze^2}{r_1} - \frac{Ze^2}{r_2} + \frac{e^2}{r_{12}}$$

where $Z = 2$, $r_1 \equiv |\vec{r}_1|$, $r_2 \equiv |\vec{r}_2|$, and $r_{12} \equiv |\mathbf{r_1} - \mathbf{r_2}|$. For the ground state, both electrons are in the $1s$ state. What are the ground-state spatial wavefunction and energy if the electron self-interaction (the last term in the above Hamiltonian) is ignored? Calculate first-order correction to the ground-state energy from the electron self-interaction. The hydrogenic ground-state wavefunction and energy are

$$\psi_{100}(\vec{r}) = \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a}\right)^{3/2} e^{-Zr/a}, \quad E_{100} = -\frac{Z^2 e^2}{2a}$$

where a is the Bohr radius. You may find the following integrals useful:

$$\int x e^{\beta x} dx = \frac{\beta x - 1}{\beta^2} e^{\beta x}; \quad \int x^2 e^{\beta x} dx = \frac{\beta^2 x^2 - 2\beta x + 2}{\beta^3} e^{\beta x}.$$

7. All hadrons except maybe the proton are unstable particles. Give an example of a specific decay of a hadron via each of the basic interactions: strong, electromagnetic, and weak. For each give a qualitative estimate of the lifetime of the decaying particle.

8. **Fermi surface of a two-dimensional electron gas:** The density of states for free spin-1/2 fermions in two dimensions in a box of area A is

$$n(\epsilon) = \frac{Am}{\pi\hbar^2},$$

independent of energy ϵ, where m is the mass of a fermion.

(a) Write an integral expressing the number N of particles in such a box in terms of the density of states and the chemical potential and hence show that the chemical potential is given by

$$\mu = k_B T \ln\left[\exp\left(\frac{\pi\hbar^2\rho}{mk_BT}\right) - 1\right],$$

where $\rho = N/A$ is the number density of fermions and k_B is Boltzmann's constant. You will need the result that

$$\int_0^\infty \frac{dx}{ae^x + 1} = \ln(1 + a^{-1}).$$

(b) What value does this give for μ as $T \to 0$?

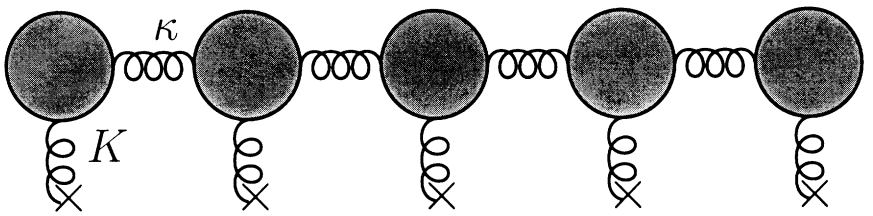
(c) As temperature rises from absolute zero, which way does the chemical potential move – up or down?

[More space for problem 8 if needed]

Part B: Optional Problems

9. An energetic cosmic ray photon will collide with the cosmic microwave background photons and produce an electron-positron pair. The CMB photons have a temperature of about 3 Kelvin. Thus few photons should be seen with energies above that needed to produce the electron-positron pair. Estimate that energy.

10. Consider a linear chain of atoms interacting with springs of spring constant κ but also with *fixed* lattice points with lattice constant a, with another spring constant K.



Find and plot the dispersion relation. You should find that ω_k does not go to zero as k goes to 0.

11. Consider the ground state and the first two excited states of potassium ($Z = 19$). Draw the energy level diagram in the presence of a small applied magnetic field and show the allowed dipole transitions.

12. ${}^{60}_{27}Co$ is a radiactive nucleus that has a half-life of 5.27 years. A sample ${}^{60}_{27}Co$ initially has 1.8×10^{16} ${}^{60}_{27}Co$ nuclei.

(a) What is its activity initially?

(b) How many nuclei remain after 26.3 years?

(c) If it decays by β^- emission, what is the atomic weight A and the atomic number Z after the decay?