There are five (5) problems in this examination.
Problem 1

Let the $A$ be a 2-by-2 matrix with real entries, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The range of $A$ is $\{Ax : x \in \mathbb{R}^2\}$.

(a) (8 points) List all possible values of the rank (the dimension of the range) of $A$. For each value of the rank, find necessary and sufficient conditions on $a, b, c, \text{ and } d$ such that $A$ has that rank.

(b) (12 points) Now for each value of the rank, find necessary and sufficient conditions on $a, b, c, \text{ and } d$ such that $A$ has that rank and $A$ is a projection matrix, i.e. $\forall v \in \text{range of } A, \; Av = v$. 
Problem 1
Problem 1
Problem 1
Problem 2

(a) (10 points) Compute the determinant of $B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 2 & 3 & 4 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 5 & 4 \end{pmatrix}$.

(b) (10 points) Find the solution to $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ which lies closest to the origin.
Problem 2
Problem 2
Problem 2
Problem 3

(a) (10 points) Find the eigenvalues and eigenfunctions of

$$y'' - 4\lambda y' + 4\lambda^2 y = 0$$  \hspace{1cm} (1)

with the boundary conditions

$$y(0) = 0, \quad y(1) + y'(1) = 0.$$  \hspace{1cm} (2)

In other words, find all solutions $y(x)$ other than the zero function, and corresponding values of the constant $\lambda$.

(b) (10 points) Find a linear scalar ODE with constant coefficients of the least possible order which has $x_1(t) = 1$ and $x_2(t) = \cos t$ as particular solutions.
Problem 3
Problem 3
Problem 3
The functions $t$, $t^5$, and $|t|^5$ are solutions to the differential equation $t^2 x'' - 5tx' + 5x = 0$.

(a) (5 points) Are the solutions linearly independent on $-1 < t < 1$?

(b) (15 points) For which initial conditions and on which intervals do we have unique solutions to the equation? In each case, what is the form of the solution?
Problem 4
Problem 4
Problem 4
Problem 5

(a) (10 points) Find the eigenvalues and eigenfunctions of the Laplacian on the square with periodic boundary conditions:

\[ u_{xx} + u_{yy} + \lambda u = 0 \quad 0 \leq x \leq 2\pi, 0 \leq y \leq 2\pi \]  
\[ u(0, y) = u(2\pi, y) \quad 0 \leq y \leq 2\pi \]  
\[ u(x, 0) = u(x, 2\pi) \quad 0 \leq x \leq 2\pi \]  

(b) (10 points) Solve the heat equation on the square with periodic boundary conditions

\[ u_t = u_{xx} + u_{yy} \quad 0 \leq x \leq 2\pi, 0 \leq y \leq 2\pi \]  
\[ u(t, 0, y) = u(t, 2\pi, y) \quad 0 \leq y \leq 2\pi \]  
\[ u(t, x, 0) = u(t, x, 2\pi) \quad 0 \leq x \leq 2\pi \]  

and initial data \( u(0, x, y) = f(x, y) \).
Problem 5
Problem 5
Problem 5